PHYSICS

Introduction to Mechanics

WITH LAB MANUAL

A.B. Bhattacharya | Atanu Nag

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FOREWORD

Engineering has played a very significant role in the progress and expansion of mankind and society for centuries. Engineering ideas that originated in the Indian subcontinent have had a thoughtful impact on the world.

All India Council for Technical Education (AICTE) had always been at the forefront of assisting Technical students in every possible manner since its inception in 1987. The goal of AICTE has been to promote quality Technical Education and thereby take the industry to a greater heights and ultimately turn our dear motherland India into a Modern Developed Nation. It will not be inept to mention here that Engineers are the backbone of the modern society - better the engineers, better the industry, and better the industry, better the country.

NEP 2020 envisages education in regional languages to all, thereby ensuring that each and every student becomes capable and competent enough and is in a position to contribute towards the national growth and development.

One of the spheres where AICTE had been relentlessly working from last few years was to provide high-quality moderately priced books of International standard prepared in various regional languages to all it's Engineering students. These books are not only prepared keeping in mind it's easy language, real life examples, rich contents and but also the industry needs in this everyday changing world. These books are as per AICTE Model Curriculum of Engineering & Technology – 2018.

Eminent Professors from all over India with great knowledge and experience have written these books for the benefit of academic fraternity. AICTE is confident that these books with their rich contents will help technical students master the subjects with greater ease and quality.

AICTE appreciates the hard work of the original authors, coordinators and the translators for their endeavour in making these Engineering subjects more lucid.

(Anil D. Sahasrabudhe)

Acknowledgement

The author(s) are grateful to AICTE for their meticulous planning and execution to publish the technical book for Engineering and Technology students.

We sincerely acknowledge the valuable contributions of the reviewer of the book Prof. R.P Dahiya, for making it students' friendly and giving a better shape in an artistic manner.

This book is an outcome of various suggestions of AICTE members, experts and authors who shared their opinion and thoughts to further develop the engineering education in our country.

It is also with great honour that we state that this book is aligned to the AICTE Model Curriculum and in line with the guidelines of National Education Policy (NEP) -2020. Towards promoting education in regional languages, this book is being translated in scheduled Indian regional languages.

Acknowledgements are due to the contributors and different workers in this field whose published books, review articles, papers, photographs, footnotes, references and other valuable information enriched us at the time of writing the book.

Finally, we like to express our sincere thanks to the publishing house, M/s. Khanna Book Publishing Company Private Limited, New Delhi, whose entire team was always ready to cooperate on all the aspects of publishing to make it a wonderful experience.

A. B. Bhattacharya, Atanu Nag

Preface

The book titled "**Physics - Introduction to Mechanics**" is an outcome of the rich experience of our teaching of basic physics courses. The initiation of writing this book is to expose basic science to the engineering students to the fundamentals of physics as well as enable them to get an insight of the subject. Keeping in mind the purpose of wide coverage as well as to provide essential supplementary information, we have included the topics recommended by AICTE, in a very systematic and orderly manner throughout the book. Efforts have been made to explain the fundamental concepts of the subject in the simplest possible way.

During the process of preparation of the manuscript, we have considered the various standard text books and accordingly we have developed sections like critical questions, solved and supplementary problems etc. While preparing the different sections emphasis has also been laid on definitions and laws and also on comprehensive synopsis of formulae for a quick revision of the basic principles. The book covers all types of medium and advanced level problems and these have been presented in a very logical and systematic manner. The gradations of those problems have been tested over many years of teaching a wide variety of students.

Apart from illustrations and examples as required, we have enriched the book with numerous solved problems in every unit for proper understanding of the related topics. Under the common title "Physics" there is a set of four books covering different aspects and applications of physics in engineering. Out of those, the first one covers Introduction to Electromagnetic Theory, the second one is based on Introduction to Mechanics, the third one is related to Quantum Mechanics for Engineers and the fourth one is based on Oscillations, Waves and Optics. It is important to note that in all the books, we have included the relevant laboratory practical. In addition, besides some essential information for the users under the heading "Know More" we have clarified some essential basic information in the appendix and annexure section.

As far as the present book is concerned, "Physics - Introduction to Mechanics" is meant to provide a thorough grounding in applied physics on the topics covered. This part of the physics book will prepare engineering students to apply the knowledge of Mechanics to tackle 21st century and onward engineering challenges and address the related aroused questions. The subject matters are presented in a constructive manner so that an Engineering degree prepares students to work in different sectors or in national laboratories at the very forefront of technology.

We sincerely hope that the book will inspire the students to learn and discuss the ideas behind basic principles of engineering physics and will surely contribute to the development of a solid foundation of the subject. We would be thankful to all beneficial comments and suggestions which will contribute to the improvement of the future editions of the book. It gives us immense pleasure to place this book in the hands of the teachers and students. It was indeed a big pleasure to work on different aspects covering in the book.

A. B. Bhattacharya, Atanu Nag

Outcome Based Education

For the implementation of an outcome based education the first requirement is to develop an outcome based curriculum and incorporate an outcome based assessment in the education system. By going through outcome based assessments evaluators will be able to evaluate whether the students have achieved the outlined standard, specific and measurable outcomes. With the proper incorporation of outcome based education there will be a definite commitment to achieve a minimum standard for all learners without giving up at any level. At the end of the programme running with the aid of outcome based education, a students will be able to arrive at the following outcomes:

PO-1: **Engineering knowledge:** Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

PO-2: **Problem analysis:** Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

PO-3: **Design/development of solutions:** Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.

PO-4: **Conduct investigations of complex problems:** Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

PO-5: **Modern tool usage:** Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.

PO-6: **The engineer and society:** Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

PO-7: **Environment and sustainability:** Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

PO-8: **Ethics:** Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

PO-9: **Individual and team work:** Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

PO-10: Communication: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

PO-11: Project management and finance: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

PO-12: Life-long learning: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

Course Outcomes

After completion of the course the students will be able to:

CO-1: Describe planar, spatial and dynamical motion of rigid bodies with the help of Euler's equations

CO-2: Explain rotational motion characteristics from the idea of torque and angular momentum

CO-3: Apply vector calculus and Newton's laws to fundamental level problems of mechanics

CO-4: Apply Kepler's laws with the aid of other conservation laws to formulate central force problems

CO-5: Apply the concepts of Coriolis force in different environmental and geographical aspects

CO-6: Analyze different types of oscillatory and vibratory motion problems in real life

Course Outcomes	Expected Mapping with Programme Outcomes *(1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation)*											
	PO-1	**PO-2**	**PO-3**	**PO-4**	**PO-5**	**PO-6**	**PO-7**	**PO-8**	**PO-9**	**PO-10**	**PO-11**	**PO-12**
CO-1	3	2	2	2	1	-	-	-	-	-	-	-
CO-2	3	1	2	1	-	-	-	-	-	-	-	-
CO-3	3	3	2	1	-	-	-	-	-	-	-	-
CO-4	3	3	3	2	1	-	-	-	-	-	-	-
CO-5	3	3	2	1	1	-	-	-	-	-	-	-
CO-6	3	3	3	2	1	-	-	-	-	-	-	-

Abbreviations and Symbols

List of Abbreviations

General Terms			
Abbreviations	**Full form**	**Abbreviations**	**Full form**
CO	Course Outcome	LCR	Inductor Capacitor Resistor
CG	Centre of Gravity	MI	Moment of Inertia
CM	Centre of Mass	PE	Potential Energy
Div	Divergence	PO	Programme Outcome
Grad	Gradient	SHM	Simple Harmonic Motion
KE	Kinetic Energy	UO	Unit Outcome
Units Used			
Abbreviations	**Full form**	**Abbreviations**	**Full form**
Hz	hertz	N	newton
J	joule	rad	radian
kg	kilogram	rps	rotation per second

List of Symbols

Symbols	Description	Symbols	Description
I	moment of inertia	μ_k	coefficient of kinetic friction
K	radius of gyration	α	angular acceleration
T	time period	θ	angular displacement
S	sharpness of resonance	λ	latitude
G	Universal gravitational constant	μ	reduced mass
U_{eff}	effective potential energy	ε	eccentricity
K_{eff}	effective kinetic energy	τ	torque
F_c	centripetal force	δ_{xy}	kronecker delta
F_g	centrifugal force	$\vec{A}.\vec{B}$	$\vec{A}$ dot $\vec{B}$
$\vec{L}$	angular momentum	$\vec{A}\times\vec{B}$	$\vec{A}$ cross $\vec{B}$
$\overleftrightarrow{I}$	moment of inertia tensor	$\vec{\nabla}$	del operator
g	acceleration due to gravity	$\vec{\nabla}V$	grad V
k	spring constant	$\vec{\nabla}.\vec{A}$	divergence $\vec{A}$
f_c	centripetal acceleration	$\vec{\nabla}\times\vec{A}$	curl $\vec{A}$
$\vec{\omega}$	angular velocity	$\nabla^2\varphi$	laplacian φ
μ_s	coefficient of static friction		

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Unit 1 Introductory Mechanics

Unit 2 Conservation Principles

Unit 3 Dynamics of Particles

Unit 4 Oscillations

Unit 5 Rotational Motion

Unit 6 Dynamics of a Rigid Body

Guidelines for Teachers

To implement Outcome Based Education (OBE) knowledge level and skill set of the students should be enhanced. Teachers should take a major responsibility for the proper implementation of OBE. Some of the responsibilities (not limited to) for the teachers in OBE system may be as follows:

- Within reasonable constraint, they should manoeuvre time to the best advantage of all students.
- They should assess the students only upon certain defined criterion without considering any other potential ineligibility to discriminate them.
- They should try to grow the learning abilities of the students to a certain level before they leave the institute.
- They should try to ensure that all the students are equipped with the quality knowledge as well as competence after they finish their education.
- They should always encourage the students to develop their ultimate performance capabilities.
- They should facilitate and encourage group work and team work to consolidate newer approach.
- They should follow Blooms taxonomy in every part of the assessment.

Bloom's Taxonomy

Level	Teacher should Check	Student should be able to	Possible Mode of Assessment
Create	Students ability to create	Design or Create	Mini project
Evaluate	Students ability to justify	Argue or Defend	Assignment
Analyse	Students ability to distinguish	Differentiate or Distinguish	Project/Lab Methodology
Apply	Students ability to use information	Operate or Demonstrate	Technical Presentation/ Demonstration
Understand	Students ability to explain the ideas	Explain or Classify	Presentation/Seminar
Remember	Students ability to recall (or remember)	Define or Recall	Quiz

Guidelines for Students

Students should take equal responsibility for implementing the OBE. Some of the responsibilities (not limited to) for the students in OBE system are as follows:

- Students should be well aware of each UO before the start of a unit in each and every course.
- Students should be well aware of each CO before the start of the course.
- Students should be well aware of each PO before the start of the programme.
- Students should think critically and reasonably with proper reflection and action.
- Learning of the students should be connected and integrated with practical and real life consequences.
- Students should be well aware of their competency at every level of OBE.

CONTENTS

1 Introductory Mechanics

UNIT SPECIFICS

Through this unit we have discussed the following aspects:

- Transformation of scalars and vectors under rotation transformation;
- Forces in nature;
- Newton's laws and its completeness in describing particle motion;
- Form invariance of Newton's second law;
- Solving Newton's equations of motion in polar coordinates;
- Problems including constraints and friction;
- Extension to cylindrical and spherical coordinates.

The practical applications of the topics are discussed for generating further curiosity and creativity as well as improving problem solving capacity.

Besides giving a large number of multiple choice questions as well as questions of short and long answer types marked in two categories following lower and higher order of Bloom's taxonomy, assignments through a number of numerical problems, a list of references and suggested readings are given in the unit so that one can go through them for practice. It is important to note that for getting more information on various topics of interest some QR codes have been provided in different sections which can be scanned for relevant supportive knowledge.

After the related practical, based on the content, there is a "Know More" section. This section has been carefully designed so that the supplementary information provided in this part becomes beneficial for the users of the book. This section mainly highlights the initial activity, examples of some interesting facts, analogy, history of the development of the subject focusing the salient observations and finding, timelines starting from the development of the concerned topics up to the recent time, applications of the subject matter for our day-to-day real life or/and industrial applications on variety of aspects, case study related to environmental, sustainability, social and ethical issues whichever applicable, and finally inquisitiveness and curiosity topics of the unit.

RATIONALE

This fundamental unit on mechanics helps students to get a primary idea about the transformation of scalars and vectors under rotation transformation and forces in nature. It explains Newton's laws and its completeness in describing particle motion. All these basic aspects are relevant to start the mechanics properly. It then explains clearly Newton's laws and its completeness in describing particle motion and the form invariance of Newton's second law as well as Newton's equations of motion in polar coordinates. All these are discussed at length to develop the subject. Some related problems are pointed out with an extension to cylindrical and spherical coordinates which can help further for getting a clear idea of the concern topics on mechanics.

Mechanics is an important branch of physical science that essentially deals with forces and energy and their effect on bodies. Mechanics started its journey by quantifying motion and then explaining it in terms of forces, energy and momentum. This permits one to analyze the operation of many day-to-day familiar phenomena around us. But at the same time it covers the mechanics of planets, stars and galaxies. Its practical applications are related to the construction, design, and operation of different type of machines and tools.

PRE-REQUISITES

Mathematics: Co-ordinate Systems (Class XII)

Physics: Mechanics (Class XII)

UNIT OUTCOMES

List of outcomes of this unit is as follows:

U1-O1: Describe basic vector operations

U1-O2: Describe the Cartesian, spherical and cylindrical coordinate system

U1-O3: Explain Newton's laws in describing particle motion

U1-O4: Realize the role of friction in constrained motion of a body

U1-O5: Apply vector calculus to solve complex problems

Unit-1 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (*1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation*)					
	CO-1	**CO-2**	**CO-3**	**CO-4**	**CO-5**	**CO-6**
U1-O1	3	3	3	-	3	1
U1-O2	1	1	2	2	1	-
U1-O3	2	1	3	1	2	1
U1-O4	-	-	3	1	2	2
U1-O5	3	3	3	-	3	1

1.1 VECTOR FUNDAMENTALS

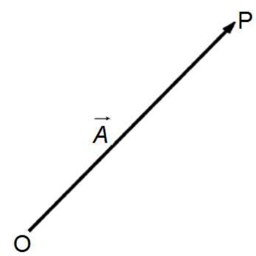


Fig. 1.1: Representation of a vector

The major advantage of vector analysis lies in the fact that a vector is independent of the coordinate system. Contrary to a scalar quantity such as temperature, energy, volume; a vector represents both magnitude and direction such as displacement, velocity, force and acceleration. A vector is identified using an arrow for defining the direction. The vector magnitude is indicated by using the length of the arrow. The tail end O of the arrow is the initial point while the head P is the final point. A vector is shown by a letter with arrow on it ($\vec{A}$), and its magnitude is denoted by A (Fig. 1.1).

1.1.1 Components of a Vector

Any vector $\vec{A}$ in three dimensions can be represented with initial point at the origin O of a rectangular coordinate system. Now, let (A_x, A_y, A_z) be the rectangular coordinates of the terminal point of vector $\vec{A}$.

The vectors $A_x\hat{i}$, $A_y\hat{j}$ and $A_z\hat{k}$ are the rectangular components or simply component vectors of $\vec{A}$ in the three directions X, Y and Z respectively [Fig. 1.2 (a)].

The sum or resultant of $A_x\hat{i}$, $A_y\hat{j}$ and $A_z\hat{k}$ vector may be taken as $\vec{A}$.

So that we can write

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \tag{1.1}$$

The magnitude of $\vec{A}$ is

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{1.2}$$

We have $\vec{A} = |\vec{A}|\hat{n}$ [where $\hat{n}$ is the unit vector along $\vec{A}$] (1.3)

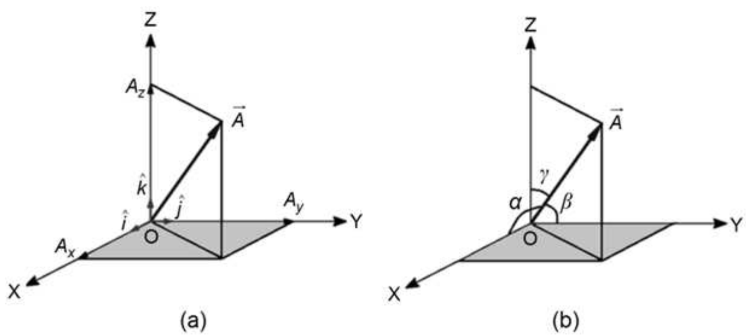


Fig. 1.2: Resolution of a vector in three dimensions

$$\therefore \quad \hat{n} = \frac{\vec{A}}{|\vec{A}|} \tag{1.4}$$

or, $$\hat{n} = \frac{A_x \hat{i} + A_y \hat{j} + A_z \hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = \frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \hat{i} + \frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \hat{j} + \frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \hat{k}$$

$\therefore$ $$\left|\hat{n}\right| = \sqrt{\left(\frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}}\right)^2 + \left(\frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}}\right)^2 + \left(\frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}\right)^2} = 1 \tag{1.5}$$

Thus it is shown that the magnitude of unit vector becomes unity, as it should be.

EXAMPLE 1.1

Example 1.1 For a unit vector $\hat{n} = \frac{1}{2}\hat{i} + m\hat{j} + \frac{1}{3}\hat{k}$; calculate the value of m.

Solution

For the unit vector $\hat{n} = \frac{1}{2}\hat{i} + m\hat{j} + \frac{1}{3}\hat{k}$ we have $\left|\hat{n}\right| = 1$

$\therefore$ $$\sqrt{\left(\frac{1}{2}\right)^2 + m^2 + \left(\frac{1}{3}\right)^2} = 1$$

or, $$m = \pm\frac{1}{6}\sqrt{23}.$$

Direction cosines: Now, the cosines of the angles made by the vector $\vec{A}$ with the co-ordinate axes are called the *direction cosines* of the vector as these helps to specify the direction of $\vec{A}$ [Fig 1.2 (b)].

So, $$\cos\alpha = \frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \tag{1.6 (a)}$$

where α is the angle that $\overrightarrow{OA}$ makes with the X-axis

$$\cos\beta = \frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \tag{1.6 (b)}$$

where β is the angle that $\overrightarrow{OA}$ makes with the Y-axis

and $$\cos\gamma = \frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \tag{1.6 (c)}$$

where γ is the angle that $\overrightarrow{OA}$ makes with the Z-axis

A unit vector is considered in the same direction as the position vector $\overrightarrow{OA}$ and is given by the expression

$$\hat{n} = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma \tag{1.7}$$

or, $$\left|\hat{n}\right| = \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}$$

$\therefore \qquad \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 1$

The direction cosines of any vector $\vec{A}$ satisfy the equation.

$$\therefore \qquad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{1.8}$$

So the squares of the direction cosines, when added together, become equal to 1.

Example 1.2: If $\vec{P} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{Q} = 2\hat{i} - 2\hat{j} + \hat{k}$, Find the magnitude and direction cosines of $\vec{P} + \vec{Q}$ and $\vec{P} - \vec{Q}$.

Solution

Given $\qquad \vec{P} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{Q} = 2\hat{i} - 2\hat{j} + \hat{k}$

Thus, $\qquad \vec{P} + \vec{Q} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{P} - \vec{Q} = -\hat{i} + \hat{j}$

So, $\qquad \left|\vec{P} + \vec{Q}\right| = \sqrt{14}$ and $\left|\vec{P} - \vec{Q}\right| = \sqrt{2}$

Direction cosines of $\vec{P} + \vec{Q}$ are

$$\cos\alpha = \frac{3}{\sqrt{14}}, \cos\beta = -\frac{1}{\sqrt{14}} \text{ and } \cos\gamma = \frac{2}{\sqrt{14}}$$

or, $\qquad \alpha = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right), \beta = \cos^{-1}\left(-\frac{1}{\sqrt{14}}\right)$ and $\gamma = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right)$

Direction cosines of $\vec{P} - \vec{Q}$ are

$$\cos\alpha' = -\frac{1}{\sqrt{2}}, \cos\beta' = \frac{1}{\sqrt{2}} \text{ and } \cos\gamma = 0$$

or, $\qquad \alpha' = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right), \beta' = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ and $\gamma' = \cos^{-1}0$.

EXAMPLE 1.2

1.1.2 Vector Algebra

If we walk 4 km east and then walk 3 km north we will have gone a total of 7 km, but we will be at a distance of 5 km from our initial starting point. The mathematical tool which helps us to describe such quantities is called *vector algebra*.

Vector Addition: Vector addition is an operation to add two or more vectors.

i) Vector addition is commutative: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

ii) Vector addition is associative: $\left(\vec{A} + \vec{B}\right) + \vec{C} = \vec{A} + \left(\vec{B} + \vec{C}\right)$

iii) Vector subtraction in form of vector addition: $\vec{A} - \vec{B} = \vec{A} + \left(-\vec{B}\right)$

iv) Vector addition multiplied by a scalar number is distributive: $m\left(\vec{A} + \vec{B}\right) = m\vec{A} + m\vec{B}$

We may add vectors in two manners, graphically and algebraically.

Graphically vector addition can be defined as shown below.

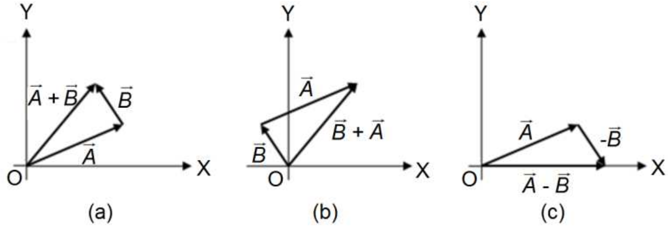


Fig. 1.3: Graphical representation of vector addition

SCAN ME
for more on vector addition and subtraction

Vector Product (multiplication): If vector $\vec{A}$ and vector $\vec{B}$ are multiplied, then the result will be either a scalar or a vector. It depends on how multiplications of those are made. The two types of vector multiplication are:

i) Scalar product (or dot product)

ii) Vector product (or cross product)

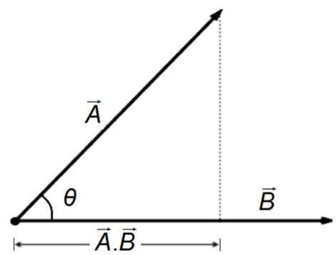


Fig. 1.4: Vector dot product

Scalar or dot product: The dot or scalar product of two vectors $\vec{A}$ and $\vec{B}$, denoted by [$\vec{A}.\vec{B}$ ($\vec{A}$ dot $\vec{B}$)], is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle θ between them (Fig. 1.4). In symbols,

$$\vec{A}.\vec{B} = AB\cos\theta \qquad (1.9)$$

Note that $\vec{A}.\vec{B}$ is a scalar and not a vector. For example force and displacement are two vectors, but their product is the work done by the force due to the displacement and is a scalar.

SCAN ME
for more on dot product

The angle between two vectors $\vec{A}$ and $\vec{B}$ is given by,

$$\theta = \cos^{-1}\left(\frac{\vec{A}.\vec{B}}{AB}\right) \qquad (1.10)$$

If the dot product of two vectors $\vec{A}$ and $\vec{B}$ is zero *i.e.*, if $\vec{A}.\vec{B} = 0$, the angle between the vectors is zero ($\theta = 0^\circ$).

EXAMPLE 1.3

Example 1.3 If $|\vec{P}+\vec{Q}| = |\vec{P}-\vec{Q}|$, then show that $\vec{P}$ is perpendicular to $\vec{Q}$.

Solution

We have $\quad |\vec{P}+\vec{Q}| = |\vec{P}-\vec{Q}|$

or, $\quad |\vec{P}+\vec{Q}|^2 = |\vec{P}-\vec{Q}|^2$

or, $\quad 2\vec{P}.\vec{Q} = -2\vec{P}.\vec{Q}$

or, $\quad 4\vec{P}.\vec{Q} = 0$

So, $\vec{P}$ and $\vec{Q}$ are perpendicular to each other.

Example 1.4 If $\hat{a}$ and $\hat{b}$ are unit vectors and θ is the angle between them, show that $2\sin\frac{\theta}{2} = |\hat{a} - \hat{b}|$.

Solution

We have $$|\hat{a} - \hat{b}|^2 = (\hat{a} - \hat{b}).(\hat{a} - \hat{b})$$

$$= |\hat{a}|^2 + |\hat{b}|^2 - 2|\hat{a}||\hat{b}|\cos\theta = 2 - 2\cos\theta = 4\sin^2\frac{\theta}{2}$$

$\therefore$ $$2\sin\frac{\theta}{2} = |\hat{a} - \hat{b}|.$$

EXAMPLE 1.4

Example 1.5 If $\vec{A}$ is a vector of constant magnitude then show that $\frac{d\vec{A}}{dt} \perp \vec{A}$, provided $\frac{d\vec{A}}{dt} \neq 0$.

Solution

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Thus, $$\vec{A}.\vec{A} = A_x{}^2 + A_y{}^2 + A_z{}^2 = \text{constant}$$

So, $$\frac{d}{dt}(\vec{A}.\vec{A}) = 2\vec{A}.\frac{d\vec{A}}{dt} = 0$$

or, $$\vec{A}.\frac{d\vec{A}}{dt} \qquad \left[\frac{d\vec{A}}{dt} \neq 0\right]$$

So, $\frac{d\vec{A}}{dt}$ is perpendicular to $\vec{A}$.

EXAMPLE 1.5

Vector or cross product: The vector or cross product of two vectors $\vec{A}$ and $\vec{B}$, denoted by [$\vec{A} \times \vec{B}$ ($\vec{A}$ cross $\vec{B}$)], is defined by

$$\vec{A} \times \vec{B} = AB\sin\theta\hat{n} \tag{1.11}$$

where θ represents the angle between the vectors. The unit vector $\hat{n}$ is in the direction normal to the plane due to the two vectors.

Now, $$|\vec{A} \times \vec{B}| = AB\sin\theta$$

Thus the unit vector is given by,

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} \tag{1.12}$$

Basically, $\vec{A}\times\vec{B}$ is a vector normal to the plane containing $\vec{A}$ and $\vec{B}$, the magnitude is given by $|\vec{A}||\vec{B}|\sin\theta$ and direction is taken following the right hand rule. It follows that $\vec{A}\times\vec{A}=0$.

The cross product is neatly written down in component form as

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

where we recall that the determinant of a 3×3 matrix is given by,

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\hat{i}\left(A_yB_z-A_zB_y\right)+\hat{j}\left(A_zB_x-A_xB_z\right)+\hat{k}\left(A_xB_y-A_yB_x\right)$$

Thus, $$\vec{A}\times\vec{B}=\hat{i}\left(A_yB_z-A_zB_y\right)+\hat{j}\left(A_zB_x-A_xB_z\right)+\hat{k}\left(A_xB_y-A_yB_x\right) \tag{1.13}$$

Unlike the dot product, the cross product is non-commutative and non-associative *i.e.*,

$$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A} \text{ and } \vec{A}\times\left(\vec{B}\times\vec{C}\right)\neq\left(\vec{A}\times\vec{B}\right)\times\vec{C}.$$

EXAMPLE 1.6

Example 1.6 If $\vec{P}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{Q}=\hat{i}-\hat{j}+2\hat{k}$, obtain a unit vector which is perpendicular to vectors $\vec{P}$ and $\vec{Q}$. Find also the angle made by the vectors.

Solution

Given vectors are $\vec{P}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{Q}=\hat{i}-\hat{j}+2\hat{k}$

$\therefore$ $$\vec{P}\times\vec{Q}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 1 & -1 & 2\end{vmatrix}=3\hat{i}-\hat{j}-2\hat{k}$$

and $$\left|\vec{P}+\vec{Q}\right|=\sqrt{3}$$

Thus the unit normal to vectors $\vec{P}$ and $\vec{Q}$ is $\hat{n}=\dfrac{\vec{P}\times\vec{Q}}{\left|\vec{P}\times\vec{Q}\right|}=\dfrac{1}{\sqrt{14}}\left(3\hat{i}-\hat{j}-2\hat{k}\right)$

If θ is the angle between $\vec{P}$ and $\vec{Q}$, then $\left|\vec{P}\times\vec{Q}\right|=\left|\vec{P}\right|\left|\vec{Q}\right|\sin\theta$,

$\therefore$ $$\sin\theta=\frac{\left|\vec{P}\times\vec{Q}\right|}{\left|\vec{P}\right|\left|\vec{Q}\right|}=\frac{\sqrt{14}}{\sqrt{3}\sqrt{4}}=\sqrt{\frac{7}{6}}$$ [as $\left|\vec{P}\right|=\sqrt{3}$ and $\left|\vec{Q}\right|=\sqrt{4}$]

or, $$\theta=\sin^{-1}\sqrt{\frac{7}{6}}.$$

Example 1.7 Given two vectors $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{B} = \hat{i} + 2\hat{j} + \hat{k}$. Find the unit vector normal to $\vec{A}$ and $\vec{B}$.

Solution

Unit vector normal to $\vec{A}$ and $\vec{B}$ is given by

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{\left|\vec{A} \times \vec{B}\right|}$$

Now,
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = -5\hat{i} + 2\hat{j} + \hat{k}$$

Thus, the unit vector normal to $\vec{A}$ and $\vec{B}$ is

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{\left|\vec{A} \times \vec{B}\right|} = \frac{-5\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{30}}.$$

EXAMPLE 1.7

Example 1.8 If $\vec{a} + \vec{b} + \vec{c} = 0$, show that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

Solution

We have $\quad \vec{a} + \vec{b} + \vec{c} = 0$

or, $\quad \vec{a} + \vec{b} = -\vec{c}$

or, $\quad \vec{b} \times \left(\vec{a} + \vec{b}\right) = -\vec{b} \times \vec{c}$

or, $\quad \vec{b} \times \vec{a} + \vec{b} \times \vec{b} = -\vec{b} \times \vec{c}$

or, $\quad -\vec{a} \times \vec{b} = -\vec{b} \times \vec{c}$

or, $\quad \vec{a} \times \vec{b} = \vec{b} \times \vec{c}$

Similarly, we can show $\vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Thus, $\quad \vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}.$

EXAMPLE 1.8

1.1.3 Vector differentiation

Del Operator: In general form the del operator ($\vec{\nabla}$) or vector differential operator is expressed as:

$$\vec{\nabla} = \frac{1}{h_1}\frac{\partial}{\partial u}\hat{a}_u + \frac{1}{h_2}\frac{\partial}{\partial v}\hat{a}_v + \frac{1}{h_3}\frac{\partial}{\partial w}\hat{a}_w \tag{1.14}$$

In *Cartesian coordinates*: $u = x,\ v = y,\ w = z;\ h_1 = h_2 = h_3 = 1$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z \tag{1.15}$$

In *cylindrical coordinates*: $u = \rho,\ v = \varphi,\ w = z;\ h_1 = h_3 = 1,\ h_2 = \rho$

$$\vec{\nabla}=\frac{\partial}{\partial\rho}\hat{a}_{\rho}+\frac{1}{\rho}\frac{\partial}{\partial\varphi}\hat{a}_{\varphi}+\frac{\partial}{\partial z}\hat{a}_{z} \tag{1.16}$$

and in *spherical polar coordinates*: $u = r,\ v = \theta,\ w = \varphi;\ h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$

$$\vec{\nabla}=\frac{\partial}{\partial r}\hat{a}_{r}+\frac{1}{r}\frac{\partial}{\partial\theta}\hat{a}_{\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\hat{a}_{\varphi} \tag{1.17}$$

Gradient of a Scalar function: We assume a scalar field V (u, v, w), which is a function of space coordinates. Gradient of this scalar field V represents a vector. We have:

$$\text{grad}\, V=\frac{\partial V}{\partial n}\hat{a}_{n}=\vec{\nabla}V \tag{1.18}$$

As ∂n is the distance along the normal and it is the shortest distance between the two surfaces (Fig. 1.5). Since the magnitude of $\dfrac{\partial V}{\partial l}=\vec{\nabla}V.\hat{a}_{l}$ depends on the direction of dl, it is named as the *directional derivative.*

When $\vec{A}=\vec{\nabla}V, V$ is named as the scalar potential function of the vector function $\vec{A}$.

EXAMPLE 1.9

Example 1.9 Find the directional derivative of $\varphi(x,y,z)=x^2y+xz^2$ at (1, 1, − 1) in the direction $\hat{i}+\hat{j}+\hat{k}$.

Solution

Here $\qquad \varphi(x,y,z)=x^2y+xz^2$

$$\therefore \quad \vec{\nabla}\varphi=\hat{i}\frac{\partial}{\partial x}\left(x^2y+xz^2\right)+\hat{j}\frac{\partial}{\partial y}\left(x^2y+xz^2\right)+\hat{k}\frac{\partial}{\partial z}\left(x^2y+xz^2\right)$$

$$=\hat{i}\left(2xy+z^2\right)+\hat{j}x^2+\hat{k}2xz$$

$$\therefore \quad \vec{\nabla}\varphi\Big|_{(11,-1)}=3\hat{i}+\hat{j}-2\hat{k}$$

The unit vector in the direction of $\hat{i}+\hat{j}+\hat{k}$ is $\hat{n}=\dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}=\dfrac{1}{\sqrt{3}}\hat{i}+\dfrac{1}{\sqrt{3}}\hat{j}+\dfrac{1}{\sqrt{3}}\hat{k}$

Hence the directional derivative of $\varphi(x,y,z)=x^2y+xz^2$ at (1, 1, − 1) in the direction $\hat{i}+\hat{j}+\hat{k}$ is

$$\vec{\nabla}\varphi\hat{n}=\left(3\hat{i}+\hat{j}-2\hat{k}\right).\left(\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}\right)=\frac{2}{\sqrt{3}}.$$

Divergence of a vector field: In vector fields, directed line segments are alternately known as flux lines or streamlines. It is the field variations as shown graphically (Fig. 1.6). The field intensity is directly proportional to the density of lines.

As for example, the number of flux lines going through a unit surface S normal to the vector determines the vector field strength. The flux of a vector field can be defined as

$$\varphi = \iint_s A\cos\theta ds = \iint_s \vec{A}.\hat{a}_n ds = \iint_s \vec{A}.ds \tag{1.19}$$

For a volume enclosed by a surface,

$$\varphi = \oint_s \vec{A}.ds \tag{1.20}$$

The divergence of a vector field $\vec{A}$ at any point P may be defined as the net outward flux from a volume enclosing P, when the volume shrinks to zero.

$$\text{div}\vec{A} = \vec{\nabla}.\vec{A} = \lim_{\Delta V \to 0} \frac{\oint_s \vec{A}.d\vec{s}}{\Delta V} \tag{1.21}$$

Here ΔV is the volume that encloses P and S is the corresponding closed surface.

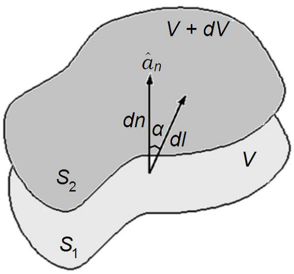


Fig. 1.5: Gradient of a scalar function

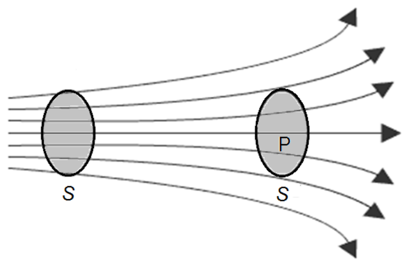


Fig. 1.6: Flux Lines

Curl of a vector field: Circulation of a vector field A in a closed path may be defined as $\oint_C \vec{A}.d\vec{l}$ and

Curl of a vector field is a measurement of the tendency to rotate about a point. Curl $\vec{A}$, also written as $\vec{\nabla} \times \vec{A}$, is defined as a vector whose magnitude is maximum of the net circulation per unit area when the area tends to zero and its direction is the normal direction to the area when the area is oriented in such a way so as to make the circulation maximum. Therefore, we can write:

$$\text{curl}\vec{A} = \vec{\nabla} \times \vec{A} = \lim_{\Delta S \to 0} \frac{\hat{a}_n}{\Delta S}\left[\oint_C \vec{A}.d\vec{l}\right]_{\max} \tag{1.22}$$

The curl of a vector function is itself a vector function. We cannot take the curl of a scalar, and in the same way we cannot take the divergence of a scalar which has no meaning.

For the geometrical interpretation of the curl, as the name implies, $\vec{\nabla} \times \vec{V}$ is a measure of how much the vector function $\vec{V}$ *curls around* the point in question.

Laplacian of a scalar field: Laplacian of a scalar function may be defined as,

$$\nabla^2\phi = \vec{\nabla}.\vec{\nabla}\phi$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi \tag{1.23}$$

EXAMPLE 1.10

Example 1.10 If $\vec{T} = \vec{F} \times \vec{R}$ and $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$, then prove that $\vec{F} = \frac{1}{2}\vec{\nabla} \times \vec{T}$

Solution

$$\vec{\nabla} \times \vec{T} = \vec{\nabla} \times \left(\vec{F} \times \vec{R}\right) = \vec{\nabla} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F_1 & F_2 & F_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{\nabla} \times \left[(F_2 z - F_3 y)\hat{i} + (F_3 x - F_1 z)\hat{j} + (F_1 y - F_2 x)\hat{k} \right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_2 z - F_3 y & F_3 x - F_1 z & F_1 y - F_2 x \end{vmatrix}$$

$$= 2\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$\therefore \qquad \vec{F} = \frac{1}{2}\vec{\nabla} \times \vec{T}.$$

EXAMPLE 1.11

Example 1.11 Given $\vec{v} = \vec{\omega} \times \vec{r}$, where $\vec{r}$ is the position vector and $\vec{\omega}$ is a constant angular velocity vector, then find out $\vec{\nabla} \times \vec{v}$.

Solution

$$\vec{\nabla} \times \vec{v} = \vec{\nabla} \times \left(\vec{\omega} \times \vec{r}\right)$$

$$= \vec{\nabla} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{\nabla} \times \left[(\omega_2 z - \omega_3 y)\hat{i} + (\omega_3 x - \omega_1 z)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k} \right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= (\omega_1 + \omega_1)\hat{i} + (\omega_2 + \omega_2)\hat{j} + (\omega_3 + \omega_3)\hat{k} = 2\vec{\omega}.$$

1.2 CO-ORDINATE SYSTEMS

To explain the spatial variations of the quantities, we need to use proper co-ordinate system. A vector can also be expressed in a curvilinear coordinate system. Let, the surfaces in the coordinate system is described by u = constant, v = constant and w = constant. In general, these surfaces are curved.

If $\hat{a}_u$, $\hat{a}_v$ and $\hat{a}_w$ be the unit vectors along the u, v and w directions, then,

$$\hat{a}_u \times \hat{a}_v = \hat{a}_w, \hat{a}_v \times \hat{a}_w = \hat{a}_u, \hat{a}_w \times \hat{a}_u = \hat{a}_v \tag{1.24}$$

Eq. (1.24) is dependent on specification of $\hat{a}_u$, $\hat{a}_v$ and $\hat{a}_w$ and also holds the following relation:

$$\hat{a}_u.\hat{a}_v = \hat{a}_v.\hat{a}_w = \hat{a}_w.\hat{a}_u = 0, \hat{a}_u.\hat{a}_u = \hat{a}_v.\hat{a}_v = \hat{a}_w.\hat{a}_w = 1 \tag{1.25}$$

In this coordinate system any vector can be represented by its orthogonal components as,

$$\vec{A} = A_u\hat{a}_u + A_v\hat{a}_v + A_w\hat{a}_w \tag{1.26}$$

Generally u, v and w do not represent length and to convert into differential changes du, dv and dw to corresponding changes dl_1, dl_2, and dl_3 in length those u, v and w are respectively multiplied by some factor of conversion h_1, h_2 and h_3. Therefore

$$d\vec{l} = dl_1\hat{a}_u + dl_2\hat{a}_v + dl_3\hat{a}_w = h_1du\hat{a}_u + h_2dv\hat{a}_v + h_3dw\hat{a}_w \tag{1.27}$$

Similarly, the differential volume dV can be expressed as $dV = h_1h_2h_3dudvdw$ and the differential surface area dS_1 normal to $\hat{a}_u$ can be expressed as $dS_1 = h_2h_3dvdw$. In a similar way, differential surface areas normal to $\hat{a}_v$ and $\hat{a}_w$ can also be defined.

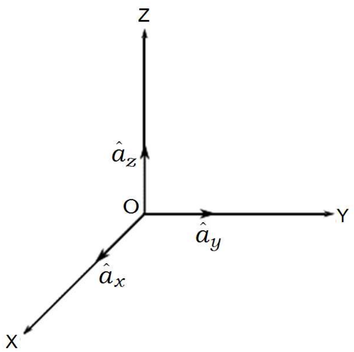


Fig. 1.7: Cartesian co-ordinate system

1.2.1 Cartesian Co-ordinate System

When considering the Cartesian coordinate system, one can write, $(u, v, w) = (x, y, z)$ [Fig. 1.7]. The unit vectors satisfy the relation given below:

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z, \hat{a}_y \times \hat{a}_z = \hat{a}_x, \hat{a}_z \times \hat{a}_x = \hat{a}_y \tag{1.28 (a)}$$

$$\hat{a}_x.\hat{a}_y = \hat{a}_y.\hat{a}_z = \hat{a}_z.\hat{a}_x = 0 \tag{1.28 (b)}$$

$$\hat{a}_x.\hat{a}_x = \hat{a}_y.\hat{a}_y = \hat{a}_z.\hat{a}_z = 1 \tag{1.28 (c)}$$

In the Cartesian co-ordinate, we have, $\vec{A} = \hat{a}_xA_x + \hat{a}_yA_y + \hat{a}_zA_z$. The dot and cross product of $\vec{A}$ and $\vec{B}$ can be expressed as follows:

$$\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z \tag{1.29}$$

and

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{a}_x\left(A_yB_z - A_zB_y\right) + \hat{a}_y\left(A_zB_x - A_xB_z\right) + \hat{a}_z\left(A_xB_y - A_yB_x\right) \tag{1.30}$$

Since x, y and z all represent lengths, $h_1 = h_2 = h_3 = 1$. The differential length, area and volume are defined respectively as

$$d\vec{l} = \hat{a}_xdx + \hat{a}_ydy + \hat{a}_zdz \tag{1.31}$$

$$d\vec{S}_x = dydz\hat{a}_x \tag{1.32 (a)}$$

$$d\vec{S}_y = dxdz\hat{a}_y \tag{1.32 (b)}$$

$$d\vec{S}_z = dxdy\hat{a}_z \tag{1.32 (c)}$$

$$dV = dxdydz \tag{1.33}$$

1.2.2 Cylindrical Co-ordinate System

For cylindrical coordinate systems $(u, v, w) = (\rho, \phi, z)$. The unit vectors satisfy the following relations:

$$\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z, \hat{a}_\phi \times \hat{a}_z = \hat{a}_\rho, \hat{a}_z \times \hat{a}_\rho = \hat{a}_\phi \tag{1.34 (a)}$$

$$\hat{a}_\rho . \hat{a}_\phi = \hat{a}_\phi . \hat{a}_z = \hat{a}_z . \hat{a}_\rho = 0 \tag{1.34 (b)}$$

$$\hat{a}_\rho . \hat{a}_\rho = \hat{a}_\phi . \hat{a}_\phi = \hat{a}_z . \hat{a}_z = 1 \tag{1.34 (c)}$$

For cylindrical coordinate we can write, $\vec{A} = \hat{a}_\rho A_\rho + \hat{a}_\phi A_\phi + \hat{a}_z A_z$.

The differential length, area and volume are defined respectively as ($h_1 = h_3 = 1$ and $h_2 = \rho$)

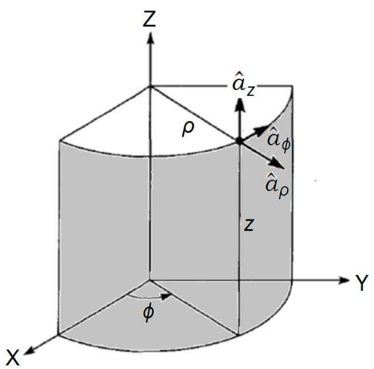


Fig. 1.8: Cylindrical coordinate system

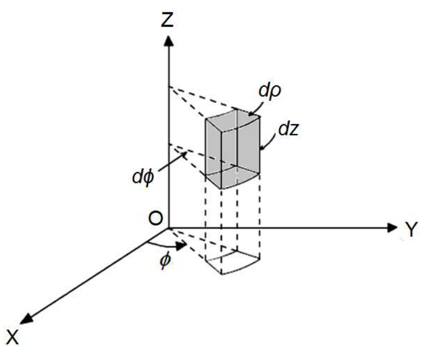


Fig. 1.9: Differential volume element in cylindrical coordinates

$$d\vec{l} = \hat{a}_\rho d\rho + \rho\hat{a}_\phi d\phi + \hat{a}_z dz \tag{1.35}$$

$$d\vec{S}_\rho = \rho d\phi dz\hat{a}_\rho \tag{1.36 (a)}$$

$$d\vec{S}_\phi = d\rho dz\hat{a}_\phi \tag{1.36 (b)}$$

$$d\vec{S}_z = \rho d\rho d\phi\hat{a}_z \tag{1.36 (c)}$$

$$dV = \rho d\rho d\phi dz \tag{1.37}$$

Transformation between Cartesian and cylindrical coordinates is:

$$x = \rho\cos\phi, y = \rho\sin\phi \text{ and } z = z \tag{1.38}$$

The inverse relationships are:

$$\rho = \sqrt{x^2 + y^2}, \phi = \tan^{-1}\frac{y}{x} \text{ and } z = z \tag{1.39}$$

1.2.3 Spherical Polar Co-ordinate System

In spherical polar coordinate we can write, $(u, v, w) = (r, \theta, \phi)$.

For the unit vectors we have,

$$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi, \hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r, \hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta \tag{1.40}$$

Fig. 1.10 exhibits the orientation of the unit vectors.

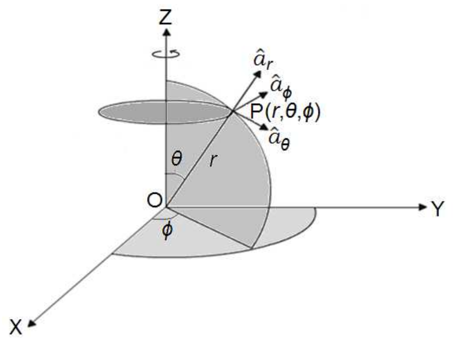


Fig. 1.10: Spherical polar coordinate system

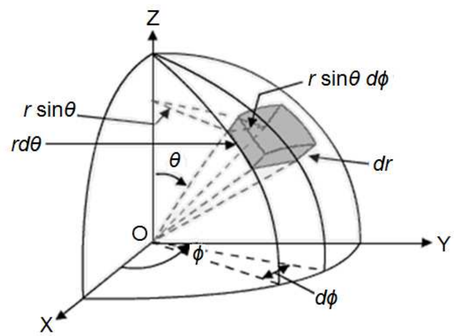


Fig. 1.11: Differential volume in spherical coordinates

For spherical polar co-ordinate, we have, $\vec{A} = \hat{a}_r A_r + \hat{a}_\theta A_\theta + \hat{a}_\phi A_\phi$.

In spherical polar coordinate we can write, $h_1 = 1$, $h_2 = r$ and $h_3 = r\sin\theta$.

With reference to Fig. 1.11, the differential length, area and volume are defined respectively as

$$d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r d\theta + \hat{a}_\phi r\sin\theta d\phi \tag{1.41}$$

$$d\vec{S}_r = r^2 \sin\theta d\theta d\phi \hat{a}_r \qquad (1.42)\ (a)$$

$$d\vec{S}_\theta = r\sin\theta dr d\phi \hat{a}_\theta \qquad (1.42)\ (b)$$

$$d\vec{S}_\phi = r dr d\theta \hat{a}_\phi \qquad (1.42)\ (c)$$

and $$dV = r^2 \sin\theta dr d\theta d\phi \tag{1.43}$$

Coordinate transformation between rectangular and spherical polar is:

$$z = x^2 + y^2$$

$$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi \text{ and } z = r\cos\theta \tag{1.44}$$

and conversely,

$$r = \sqrt{x^2 + y^2 + z^2} \qquad (1.45)\ (a)$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad (1.45)\ (b)$$

and $$\phi = \tan^{-1}\frac{y}{x} \qquad (1.45)\ (c)$$

Example 1.12 Convert the following rectangular Cartesian coordinate (1, 3, 5) into an equivalent (a) cylindrical and (b) spherical coordinates.

Solution

(a) The conversion relationships for cylindrical coordinates are:

$$\rho = \sqrt{x^2 + y^2}, \phi = \tan^{-1}\frac{y}{x} \text{ and } z = z$$

So, here $\rho = \sqrt{1^2 + 3^2} = \pm\sqrt{10} = \pm 1.73$

$\phi = \tan^{-1} 3 = 71.56^\circ$

and $z = 5$

So, the required cylindrical coordinate will be $= \left(\pm 1.73, 71.56^\circ, 5\right)$.

(b) The conversion relationships for spherical coordinates are:

$$r = \sqrt{x^2 + y^2 + z^2}, \theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} \text{ and } \phi = \tan^{-1}\frac{y}{x}$$

So, here $r = \sqrt{1^2 + 3^2 + 5^2} = \pm\sqrt{35} = \pm 5.92$

$\theta = \cos^{-1}\frac{5}{\sqrt{35}} = 32.31^\circ$

and $\phi = \tan^{-1} 3 = 71.56^\circ$

So, the required spherical coordinate will be $= \left(\pm 1.73, 32.31^\circ, 71.56^\circ\right)$.

EXAMPLE 1.12

1.3 TRANSFORMATION PROPERTIES

1.3.1 Transformation of Vector Components

Let $\vec{A}$ be any vector, having components A_x in the $\hat{e}_x$ basis and components $A_{x'}$ in the $\hat{e}_{x'}$ basis. *i.e.*,

$$\vec{A} = A_x \hat{e}_x = A_{x'} \hat{e}_{x'}$$

Both bases being orthonormal, we have,

$$A_y = \vec{A}.\hat{e}_y \text{ and } A_{y'} = \vec{A}.\hat{e}_{y'}$$

We can express the components as,

$$A_{x'} = \vec{A}.\hat{e}_{x'} = A_y \hat{e}_y \hat{e}_{x'} = \lambda_{xy} A_y$$

$$A_x = \vec{A}.\hat{e}_x = A_{z'}.\hat{e}_{z'}.\hat{e}_x = \lambda_{zx} A_{z'} = \lambda_{xz}^T A_{z'}$$

We cannot give a prime on the vector itself as there is a single vector only. However, the components of this vector are different in different bases.

In matrix form we can write these relations as

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \lambda \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \tag{1.46}$$

1.3.2 Transformation of the Scalar Product

Let $\vec{A}$ and $\vec{B}$ be vectors with components A_x and B_x in the $\hat{e}_x$ basis and components $A_{x'}$ and $B_{x'}$ in the $\hat{e}_{x'}$ basis, Now, in the $\hat{e}_x$ basis the scalar product of $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A}.\vec{B} = A_x B_x$ and in the $\hat{e}_{x'}$ basis the scalar product of $\vec{A}$ and $\vec{B}$ is denoted by

$$\left(\vec{A}.\vec{B}\right)' = A_{x'}.B_{x'} = \lambda_{xy} A_y \lambda_{xz} B_k = \delta_{jk} A_y B_k = A_y B_y = \vec{A}.\vec{B} \tag{1.47}$$

The scalar product therefore becomes same when evaluated in any basis which is desired from the geometrical interpretation of scalar product which is independent of basis.

1.3.3 Transformation of the Vector Product

Consider the inversion

$$\hat{e}_{x'} = -\hat{e}_x \qquad [\lambda_{xy} = -\delta_{xy}]$$

Now, $\quad A_{x'} = -A_x$ and $B_{x'} = B_x$

Also, $\quad A_{x'}\hat{e}_{x'} = (-A_x)\left(-\hat{e}_x\right) = A_x \hat{e}_x = \vec{A}$

so the vectors $\vec{A}$ and $\vec{B}$ are unchanged by the transformation, as they should be.

However if we calculate the vector product in the new basis using the formula

$$\vec{C}' = \begin{vmatrix} \hat{e}_{x'} & \hat{e}_{y'} & \hat{e}_{z'} \\ A_{x'} & A_{y'} & A_{z'} \\ B_{x'} & B_{y'} & B_{z'} \end{vmatrix} = (-1)^3 \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = -\vec{C} \tag{1.48}$$

The explanation is that if $\hat{e}_x$ was a R.H. basis, $\hat{e}_{x'}$ now becomes a L.H. basis as λ is an improper transformation. For the vector product the formula holds in a right-handed basis. When considered this formula for left hand basis there will be a reverse direction of the vector product. As the calculation depends on the handedness of the basis it is termed a *pseudo vector* or an *axial vector*.

Key points of suffix notation: We label basis vectors 1, 2, 3 thus $\vec{A} = \sum_{x=1}^{3} A_x \hat{e}_x$

The kronecker delta δ_{xy} can be used to define an orthonormal basis $\hat{e}_x.\hat{e}_y = \delta_{xy}$

δ_{xy} has a very useful sifting property: $\sum_y A_y \delta_{yz} = \sum_z A_z$ and is represented in matrix form by the identity matrix.

Key points of summation convention: By making use of summation convention we have, as for example,

$$\vec{A} = A_x \hat{e}_x \text{ and } A_y \delta_{yz} = A_z$$

We introduce ε_{xyz} to express the vector products of basis vectors in a uniform manner

$$\hat{e}_x \times \hat{e}_y = \varepsilon_{xyz} \hat{e}_z \tag{1.49}$$

The vector products and scalar triple products are

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

or equivalently $\left(\vec{A} \times \vec{B}\right)_x = \varepsilon_{xyz} A_y B_z$ (1.50)

and
$$\vec{A}.\vec{B} \times \vec{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

or, equivalently $\left(\vec{A}.\vec{B} \times \vec{C}\right)_x = \varepsilon_{xyz} A_x B_y C_z$ (1.51)

Key points of change of basis: The new basis is written in terms of the old through $\hat{e}_{x'} = \lambda_{xy} \hat{e}_y$ where λ is the transformation matrix λ is an orthogonal matrix, the defining property of which is

$$\lambda^{-1} = \lambda^T$$

and this can be written as $\lambda \lambda^T = 1$

or, $\lambda_{xz} \lambda_{yz} = \delta_{xy}$ (1.52)

$|\lambda| = \pm 1$ decides whether the transformation is proper or improper *i.e.* whether the handedness of the basis is changed.

Key points of algebraic approach: A scalar is defined as an invariant number in orthogonal transformation. A vector is defined as an object $\vec{A}$ represented in a basis by numbers A_x to $A_{x'}$ through $A_{x'} = \lambda_{xy} A_y$

or, in matrix form $\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \lambda \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$

1.4 FUNDAMENTAL FORCES OF NATURE

Four fundamental forces have identified in nature which are known as:

Gravitational force, Strong force, Weak force and Electromagnetic force.

Gravitational force: This is the weakest force in nature but has an infinite range. This force is always attractive and can act between any two object in nature. The effect of this gravitational force is dependent upon the mass of the two bodies and the distance between them. In comparison to the strength of the strong force it has strength of 6×10^{-39}.

Strong force: It is the strongest of the forces, but it is very short ranged. This force holds the nucleus of an atom together. It has a range of about 10^{-15} m. This is the mean diameter of a nucleus of medium sized. This force is attractive.

Weak force: This force is weak in comparison to the strong force. It has the shortest range of 10^{-18} m and this is 0.1% of the diameter of a proton. It causes the radioactive decay particularly nuclear beta decay. All particles experience this force.

Electromagnetic force: This is the second strongest force and acts on electrically charged particles. Owing to two charges this force is both attractive and repulsive. Its strength is 1/137 with respect to the strong force but the range is infinity.

1.5 NEWTON'S LAWS OF MOTION

Newton's laws of motions are as follows –

First law: Everybody maintain its state of rest or uniform motion in a straight line unless a force is applied on it.

Second law: The rate of change of momentum of a body is directly proportional to the applied force and the change occurs in the direction where the force acts.

Third law: Action and reaction are equal and opposite.

From first law of Newton we get the concepts of the following two physical quantities – (a) inertia of matter and (b) definition of force.

Inertia of a body is the property by virtue of which it opposes the cause which tends to change the state of rest or that of uniform motion of the body in a straight line.

Inertia can be classified as – (a) inertia of rest and (b) inertia of motion.

Force is basically the external system that alters or tends to alter the state of rest or of uniform motion of a body on which it acts. If external force is not applied then the body at rest will continue its status and similarly a body in motion with uniform velocity continues its motion along the straight line.

Force possesses both magnitude and direction and is the external agent which cause acceleration or retardation of the body.

Dimension of force is MLT^{-2}. SI unit of force is newton (N) and its CGS unit is dyne. 1 N $=10^5$ dyne.

1.5.1 Momentum

When a body is in motion or rest then to change its state of rest or motion the force to be applied on it which depends on – (a) the mass of the body and (b) the velocity of the body.

This helps Newton to introduce a new quantity, called linear momentum of the body and may be defined as the product of the mass (m) of the body and its velocity (v).

Thus, momentum $\vec{p} = m\vec{v}$ (1.53)

It is a vector quantity and is has the same direction as that of the linear velocity.

From Newton's second law it can be stated that when a total force $\vec{F}$ is applied to a particle, its momentum $\vec{p}$ changes at the rate $\frac{d\vec{p}}{dt}$. Mathematically, it can be represented as,

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d\left(m\vec{v}\right)}{dt} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2} = m\vec{a} \qquad (1.54)$$

where m is the mass of the particle and $\vec{a}$ is the acceleration of the particle.

Dimension and unit of momentum: Dimension of momentum is $\left[p\right] = \left[mv\right] = \left[MLT^{-1}\right]$

SI unit of momentum is kg-ms^{-1} and CGS unit of momentum is gm-cms^{-1}.

EXAMPLE 1.13

Example 1.13 Show that the momentum of a body having mass m and kinetic energy E is $\sqrt{2mE}$.

Solution

Kinetic energy $E = \frac{1}{2}mv^2$, where v = velocity

or, $v^2 = \frac{2E}{m}$

or, $v = \sqrt{\frac{2E}{m}}$

or, $mv = m\sqrt{\frac{2E}{m}} = \sqrt{2mE}$

Since mv = momentum, therefore momentum for body is $\sqrt{2mE}$.

Principle of conservation of linear momentum: It states that when no external force acts on a system of particles, the linear momentum remains conserved.

Explanation: If two masses m_1 and m_2 having initial velocities u_1 and u_2 collide with each other such that their final velocities are v_1 and v_2 then for the conservation of linear momentum we must have,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \qquad (1.55)$$

Thus, the total linear momentum before collision = that after collision.

Establishment of laws of conservation of linear momentum from Newton's third law: For the mutual interaction of two particles of masses m_1 and m_2 moving with initial velocities u_1 and u_2 for a time t, let the final velocities are v_1 and v_2. Now at the time of collision the impulsive force acting on the first body should be equal and opposite to that acting on the second body (according to Newton's third law) *i.e.*, $F_1 = -F_2$

Now if the time of action is t then the change in momentum of the first body is $= F_1t = m_1v_1 - m_1u_1$ and the change in momentum of the second body is $F_2t = m_2v_2 - m_2u_2.$

$\because \qquad F_1 = -F_2$

$$\therefore \qquad m_1v_1 - m_1u_1 = -(m_2v_2 - m_2u_2)$$

$$\text{or,} \qquad m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \qquad (1.56)$$

So, the total linear momentum of the bodies before and after the collision is same and this is the law of conservation of linear momentum.

1.5.2 Recoil of a Gun

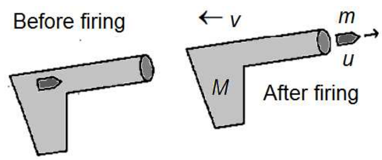


Fig. 1.12: Recoil of a gun

When a gun fires a shell it is moves forward with a large velocity and the gun moves backward with a velocity called the *recoil velocity* of the gun. This recoil velocity is due to the law of conservation of linear momentum.

Let, M = mass of empty gun, m = mass of the shell, V = velocity of the gun just after firing and v = velocity of the shell when fired from the gun (Fig. 1.12).

Now, before firing the total linear momentum of the gun and the shell is zero (0) and just after firing the total linear momentum of the gun and the shell becomes ($MV + mv$).

According to the law of conservation of linear momentum, $MV + mv = 0$

$$\text{or,} \qquad V = -\frac{mv}{M} \qquad (1.57)$$

Eq. (1.57) gives the expression for recoil velocity of the gun. The negative sign in this equation indicates that the velocity of the gun and that of the shell are opposite to each other.

Example 1.14 The mass of a machine gun is 8 kg. It fires 30 gm bullet at the rate of 6 bullets/s with a speed of 400 m/s. What force must be applied to the gun to keep it in position?

Solution

Given, mass of bullet (m) = 30 gm = 30×10^{-3} kg, speed of bullet (v) = 400 m/s, mass of the gun (M) = 8 kg and number of bullet fires per sec (n) = 6.

Let, V is the recoil velocity of the gun just after firing the bullet.

Then according to the principle of timer momentum, $MV + mu = 0$

or, $\qquad MV = -mu$

Hence, the backward momentum imparted to the gun per sec is $F = nMV = -nmu$

or, $\qquad F = -6 \times 30 \times 10^{-3} \times 400 \text{ N} = -72 \text{ N}$

Thus, the forward force required to be applied to the gun to keep it in position is 72 N.

EXAMPLE 1.14

1.6 MOTION OF TWO CONNECTED BODIES

1.6.1 Horizontal Motion of Two Bodies

Let two masses m_1 and m_2 are connected by an inextensible light string and is kept on a smooth horizontal table (Fig. 1.13). Let a horizontal force F is acting on the mass m_2, when it produces an acceleration f of the system.

If T be the tension in the string then the equation of motion of mass m_1 will be,

$$T = m_1 f \qquad \text{(i)}$$

and that of mass m_2 is

$$F - T = m_2 f \qquad \text{(ii)}$$

Adding (i) and (ii) we get,

$$F = (m_1 + m_2) f$$

$$\therefore \qquad f = \frac{F}{m_1 + m_2} \qquad (1.58)$$

$$\text{and,} \qquad T = m_1 f = \frac{m_1 F}{m_1 + m_2} \qquad (1.59)$$

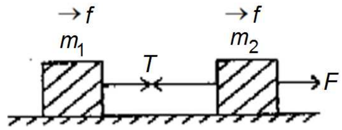


Fig. 1.13: Motion of two bodies connected by inextensible weightless string

1.6.2 Vertical Motion of Two Bodies

Let two masses m_1 and m_2 are connected by a string subjected to a vertical force to produce an upward acceleration f (Fig. 1.14). If T be the tension on the string then the net force on mass m_1 is

$$F - m_1 g - T = m_1 f \qquad \text{(i)}$$

and the net force on mass m_2 is,

$$T - m_2 g = m_2 f \qquad \text{(ii)}$$

Adding (i) and (ii) we get,

$$F - m_1 g - m_2 g = (m_1 + m_2) f$$

$$\text{or,} \qquad F = (m_1 + m_2)(g + f) \qquad (1.60)$$

and from (ii) we get

$$T = m_2 (g + f) \qquad (1.61)$$

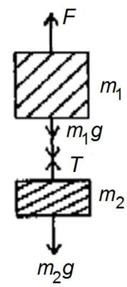


Fig. 1.14: Vertical motion of two bodies connected by inextensible string

1.7 CONSERVATION OF MECHANICAL ENERGY

The sum of the kinetic energy (KE) and potential energy (PE) of a body is its total mechanical energy. If there is no dissipative force then the total KE and PE of a body remains constant and this is the law of conservation of mechanical energy.

1.7.1 Freely Falling Body

Let a body of mass m is at rest at the point P as shown in Fig. 1.15. Let the height of the point P from the reference plane AB is h. The energy of the body is zero at the reference plane and is maximum (mgh) at point P. Now, let the body falls freely such that its velocity at point Q at a height $(h - x)$ from the reference plane is v.

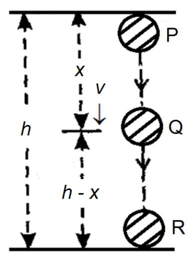


Fig. 1.15: Total mechanical energy of a freely falling body

Total mechanical energy at P

PE at point P = mgh

and KE at point P = 0.

$\therefore$ Total mechanical energy at point P = $mgh + 0 = mgh$.

SCAN ME

for better understanding conservation of energy

Total mechanical energy at Q

If v be the velocity at point Q then the KE at point Q is

$$= \frac{1}{2} mv^2 = \frac{1}{2} m \times 2gx = mgx$$

$$(\because v^2 = 0^2 + 2gx = 2gx)$$

and PE at point Q is = $mg\,(h - x)$.

$\therefore$ Total mechanical energy at point Q = $mgx + mg\,(h - x) = mgh$.

Total mechanical energy at R

PE at point R is 0

and KE at point R is = $\frac{1}{2} mv^2 = \frac{1}{2} m \times 2gh = mgh$.

$\therefore$ Total mechanical energy at R = $mgh + 0 = mgh$.

Thus the total mechanical energy at P = total mechanical energy at Q = total mechanical energy at R.

So, the total mechanical energy of a freely falling body remains conserved and this is the law of conservation of mechanical energy.

1.7.2 Body Falling Down an Inclined Plane

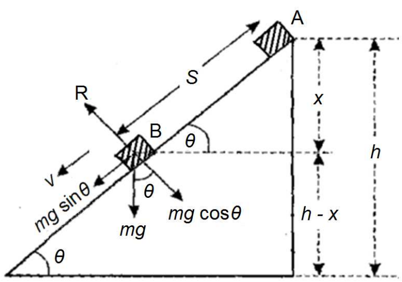


Fig. 1.16: Total mechanical energy of body falling down an inclined plane

Let a body of mass m is rest at point A on an inclined plane of inclination θ (Fig. 1.16). Now the KE of the body at A is 0 and the PE is = mgh.

$\therefore$ Total mechanical energy at A = $0 + mgh = mgh$.

When the body is left to fall along the inclined plane then let the velocity of the body is v at a distance S from A at point B. In this time if the displacement of the body along the vertical is x then from $v^2 = u^2 + 2fS$ we get,

$$v^2 = 0^2 + 2g \sin\theta \times S = 2gx \quad [\text{since, } x = S \sin\theta]$$

So, the KE of the body at B is

$$\frac{1}{2} mv^2 = \frac{1}{2} m.2gx = mgx$$

and PE at B = $mg\,(h - x)$.

$\therefore$ Total mechanical energy at point B = $mgx + mg\,(h - x)$

$$= mgh.$$

So, the total mechanical energy at A = total mechanical energy at B.

Thus the total mechanical energy of a body falling down a frictionless inclined plane remains conserved.

SCAN ME

for calculating the components of weight

1.8 FRICTION

Friction is a resistive force to the direction of motion. The frictional force is proportional to the normal force *N*. It is perpendicular to the surface between two objects. The proportionality constant is named as the coefficient of friction μ such that,

$$\mu = \frac{F}{N} \tag{1.62 (a)}$$

where *F* is the resistive force (frictional force) acts along the direction of motion

There are two coefficients of friction where the difference depends on if the object is in motion or at rest. The *coefficient of static friction* is used at the static condition. If the block is in motion, then we use the *coefficient of kinetic friction*.

1.8.1 Laws of Friction

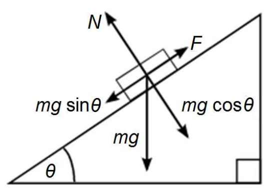


Fig. 1.17: Angle of friction

First law: Force of friction is proportional to the impressed load.

Second law: Force of friction does not depend on the apparent area of contact.

Third law: Kinetic friction does not depend on the sliding velocity.

1.8.2 Angle of Friction

The *angle of friction* or *friction angle* may be defined as:

$$\tan\theta = \frac{F}{N} = \mu_s \tag{1.62 (b)}$$

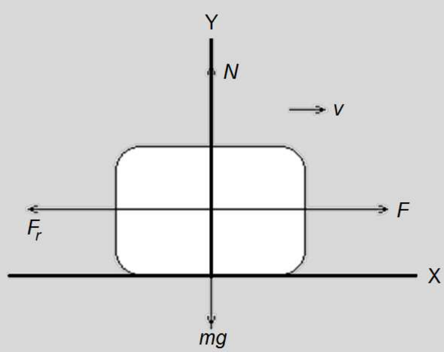
where θ is the angle from horizontal and μ_s is the static coefficient of friction between the objects.

EXAMPLE 1.15

Example 1.15 A chunk of stone of 100 kg with constant velocity of 0.5 m/s is pulled on a horizontal surface with a horizontal force of 200 N. Assume g = 9.8 m/s^2. Find the coefficient of kinetic friction.

Solution

Let F_r is the frictional force, *N* is the normal force, *mg* is the weight of the block and *F* is the force exerted to move the block. We choose a coordinate system where the horizontal right is the positive X-direction and the vertical up is the positive Y-direction. The body is in equilibrium since the velocity is constant. This means the total force acting on the block are equal to zero.



Let us consider the forces in the X-direction.

Then, $\qquad \Sigma F_x = F - F_r = 0$

So, $F = F_r$

The friction force is equal to $\mu_k N$; *i.e.*,

$$F = \mu_k N$$

Now we need to know the normal force.

We get that from the forces in the Y-direction.

$$\Sigma F_y = N - mg = 0$$

or, $N = mg$

Substituting this normal force into the previous equation, $F = \mu_k mg$

we get, $\mu_k = F/mg = 0.2$.

EXAMPLE 1.15

Example 1.16 A block weighing 200 N is pushed along a surface. When it takes 80 N for moving a block and 40 N to continue the movement of the block at a constant velocity, find the coefficients of static friction μ_s and kinetic friction μ_k?

Solution

For the coefficient of static friction, the required force to move the block is 80 N.

Now, from $F_f = \mu_s N$

where N is equal to the weight of the block = 200 N

we have, $80\ N = \mu_s \times 200\ N$

or, $\mu_s = 0.4$

For the coefficient of kinetic friction, the required force to maintain a constant velocity is 40 N.

So, from $F_f = \mu_k N$ we have,

$$40\ N = \mu_k \times 200\ N$$

or, $\mu_k = 0.2$

Thus, the two coefficients of friction for this system are $\mu_s = 0.4$ and $\mu_k = 0.2$.

EXAMPLE 1.16

1.8.3 Energy Dissipation due to Friction

When a body is moving in an inclined plane having friction then it will have to do some work against the frictional force. So, the energy gets dissipated in doing work against the frictional force and thus total mechanical energy is not conserved. Let a body of mass m is at rest at A at a height h in an inclined plane of angle of inclination θ (Fig. 1.18).

So, the PE at A = mgh and the KE at A = 0.

Thus, the total mechanical energy at A = 0 + mgh = mgh.

Now, if the body falls down along the inclined plane from A to B then the force acting on the body is mgh and let the displacement of the body due to this is S such that the velocity of the body at point B is v and is given by,

$$v^2 = u^2 + 2fS = 0 + 2fS = 2fS$$

Now, the downward force acting over the inclined plane is $mg\sin\theta - F$ where F is the frictional force acting opposite to the motion of the body.

So, the acceleration of the body along the inclined plane is $f = \dfrac{mg\sin\theta - F}{m}$

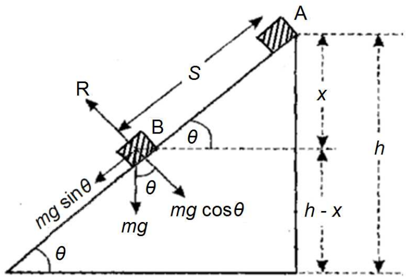


Fig. 1.18: Dissipation of energy due to friction over an inclined plane

and thus the KE of the body at $B = \dfrac{1}{2}mv^2$

$$= \frac{1}{2}m \times 2fS = mfS$$

$$= mS\,\frac{mg\sin\theta - F}{m}$$

$$= mgS\sin\theta - FS$$

$$= mgx - FS$$

Again, the PE of the body at B = $mg\,(h - x)$.

Thus, the mechanical energy of the body at B is $= mgx - FS + mg\,(h - x) = mgh - FS$.

Dissipation of energy to bring the body from point A to point B

= total mechanical energy at A – total mechanical energy at B

$= mgh - (mgh - FS) = FS$ = work done against the dissipative or frictional force F.

1.9 LIMITATIONS OF NEWTON'S LAWS

Direct applications of Newton's laws in solving dynamical problems find the following limitations:

i) Newton's laws are only valid in the Cartesian-like inertial frames of references. In case of non-inertial frames such as a frame attached to the earth rotating with it around its axis of rotation or the motion of an object with respect to the earth, a transformed version of the Newton's equations of motion is obtained incorporating the pseudo forces like the *Coriolis force* or the *centrifugal force* to make it valid in such a frame of reference.

ii) Newton's laws require the complete specification of all the forces acting on the body whose motion is under consideration.

So, the use of the Newton's laws in solving dynamical problems is only restricted to Cartesian frames with the complete knowledge of the forces acting on the body at any particular time.

1.10 INERTIAL REFERENCE FRAME

The concept of absolute rest or motion as described by Newton's law involves the concept of the proper choice of reference frame with respect to which the rest or motion of the particle is considered. In general, that frame of reference is employed in which the motion is easy to describe.

For example, motion of a train with respect to another train will be different from its motion with respect to other stationary object. Similarly the motion of a passenger in a moving bus can be described by taking the bus as the reference frame.

A reference frame which is at rest or moving with uniform speed is known as the inertial reference frame. Newton's laws are valid in this reference frame. In an inertial reference frame if a body is not experiencing a force externally, its acceleration will be zero.

If two inertial frames are moving with constant relative velocities with each other then the acceleration of a particle as experienced in both the reference frames will be identical.

1.10.1 Cartesian Coordinate System

Cartesian coordinates specify the location of a point in a plane or in a three-dimensional space. In two-dimensions the Cartesian coordinates of a point are a pair of points (x, y) which can specify the position of any particle at some instant.

Let m be the mass of a particle moving with respect to an inertial frame of reference. Let the position of the particle with reference to the origin O of the reference frame is $\vec{r}$. If $\hat{i}$, $\hat{j}$ and $\hat{k}$ be the unit vectors along X, Y and Z directions of the coordinate axes, then we can write,

$$\therefore \qquad \vec{r} = \hat{i}x + \hat{j}y + \hat{k}z \tag{1.63}$$

The velocity of the particle is thus

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\hat{i}x + \hat{j}y + \hat{k}z\right) = \hat{i}\dot{x} + \hat{j}\dot{y} + \hat{k}\dot{z} \tag{1.64}$$

For simplicity of this expression we have assumed that the three unit vectors are constant vectors. If these are not constant, an additional term will arise from the product rule.

Similarly, we can write the particle acceleration as

$$\vec{f} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\hat{i}\dot{x} + \hat{j}\dot{y} + \hat{k}\dot{z}\right) = \hat{i}\ddot{x} + \hat{j}\ddot{y} + \hat{k}\ddot{z} \tag{1.65}$$

This acceleration vector is required in Newton's second law.

Now, in terms of the components of the force along the X, Y and Z direction of the coordinate axes of the reference frame we can write

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} \tag{1.66}$$

where F_x, F_y and F_z are the components of the force along the X, Y and Z directions respectively.

Hence Newton's second law in vector form will be $\hat{i}F_x + \hat{j}F_y + \hat{k}F_z = m\left(\hat{i}\ddot{x} + \hat{j}\ddot{y} + \hat{k}\ddot{z}\right)$

Equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ we have:

$$F_x = m\ddot{x}\ ;\ F_y = m\ddot{y}\ ;\ F_z = m\ddot{z} \tag{1.67}$$

Let us take the initial conditions on the position and velocity of the particle. We also choose functional representation of the force components. Then these equations of motion can be integrated for getting the position and velocity as functions of time.

1.10.2 Polar Coordinate System

This is a two-dimensional coordinate system where all the points on a plane are found by a distance (r) with respect to a point and the angle (θ) subtended to a reference direction (Fig 1.19). Any point is thus represented in polar coordinate system as (r, θ). The Cartesian coordinates (x, y) of a point is related to its polar coordinates (r, θ).

To get a concept in two dimensions let us draw a right angled triangle as illustrated in Fig. 1.19. The hypotenuse is represented by the line segment with reference to the origin up to the point, and its length is denoted as r. The projection of this line segment on the X-axis and Y-axis are

$$x = r\cos\theta \tag{1.68}$$

and
$$y = r\sin\theta \tag{1.69}$$

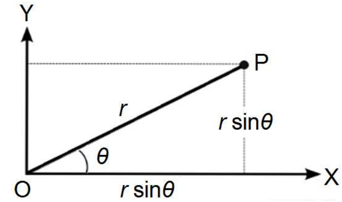


Fig. 1.19: Polar Coordinate System

The above equations give the relationship between the polar (r, θ) and the Cartesian (x, y) coordinates of a point.

The following formula can be used to convert from Cartesian to polar coordinates of any point:

$$r = \sqrt{x^2 + y^2} \qquad (1.70)\ (a)$$

and
$$\theta = \tan^{-1}\frac{y}{x} \qquad (1.70)\ (b)$$

The position vector can then be written as

$$\vec{r} = \hat{i}x + \hat{j}y = \hat{i}\,r\cos\theta + \hat{j}\,r\sin\theta \tag{1.71}$$

Differentiating once we find that the velocity as

$$\vec{v} = \frac{d}{dt}\left(\hat{i}\,r\cos\theta + \hat{j}\,r\sin\theta\right) = \hat{i}\left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right) + \hat{j}\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right) \tag{1.72}$$

Differentiating the velocity, we get the acceleration as

$$\vec{f} = \frac{d}{dt}\left[\hat{i}\left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right) + \hat{j}\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)\right]$$

$$= \hat{i}\left(\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta\right)$$

$$+\hat{j}\left(\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta\right) \tag{1.73}$$

However, note that each of the terms in the acceleration vector contains second order derivatives of both r and θ. Typically, polar coordinates are most useful when the forces are readily expressed in directions related to coordinates of a point.

1.10.3 Polar Coordinate in a Rotating Frame

Instead of using Cartesian coordinate system described by the mutually perpendicular unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ we use a new set of system with the unit vectors $\hat{e}_r$ along the radial direction, $\hat{e}_\theta$ along the cross-radial direction and $\hat{e}_z$ which is perpendicular to both $\hat{e}_r$ and $\hat{e}_\theta$ such that

$$\hat{e}_z = \hat{e}_r \times \hat{e}_\theta .$$

$\therefore$
$$\hat{e}_r = \hat{i}\cos\theta + \hat{j}\sin\theta \tag{1.74}$$

and
$$\hat{e}_\theta = \hat{i}\cos\left(\frac{\pi}{2}+\theta\right) + \hat{j}\sin\left(\frac{\pi}{2}+\theta\right) = -\hat{i}\sin\theta + \hat{j}\cos\theta \tag{1.75}$$

Then the position vector can be written as,

$$\vec{r} = r\hat{e}_r \tag{1.76}$$

Thus, the velocity vector is

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(r\hat{e}_r\right) = \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

One easily finds that

$$\frac{d\hat{e}_r}{dt} = \frac{d\left(\hat{i}\cos\theta + \hat{j}\sin\theta\right)}{dt} = \left(-\hat{i}\,\sin\theta + \hat{j}\cos\theta\right)\dot{\theta} = \dot{\theta}\hat{e}_\theta$$

Similarly, $$\frac{d\hat{e}_\theta}{dt} = \frac{d\left(-\hat{i}\sin\theta + \hat{j}\cos\theta\right)}{dt} = -\left(\hat{i}\,\cos\theta + \hat{j}\,\sin\theta\right)\dot{\theta} = -\dot{\theta}\,\hat{e}_r$$

Thus, the velocity of the particle is

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \tag{1.77}$$

Here $v_r = \dot{r}$ is the radial component of velocity and $v_\theta = r\dot{\theta}$ is the cross radial component of velocity for the motion of a particle in a rotating frame of reference.

The acceleration of the particle can be obtained by taking derivative of the velocity vector as,

$$\vec{f} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\right)$$

$$= \ddot{r}\hat{e}_r + \dot{r}\frac{d\hat{e}_r}{dt} + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$$

or, $$\vec{f} = \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\,\hat{e}_r = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta \tag{1.78}$$

Here $f_r = \ddot{r} - r\dot{\theta}^2$ = the radial component of acceleration and

$f_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ = the cross radial component of acceleration

for the motion of a particle in a rotating frame of reference. All these mathematical forms have importance in the solution of orbital mechanics.

According to Newton's second law in polar coordinates

$$\hat{e}_r F_r + \hat{e}_\theta F_\theta = m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta\right] \tag{1.79}$$

Equating the coefficients of $\hat{e}_r$ and $\hat{e}_\theta$ we have:

$$F_r = m\left(\ddot{r} - r\dot{\theta}^2\right) \tag{1.80}$$

$$F_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) \tag{1.81}$$

Similarly these can be calculated for cylindrical coordinates. So, summarizing we can have the expressions for Newton's law in various coordinate systems as shown in Table 1.1.

Table 1.1: Components of force in different coordinate systems

Cartesian	Spherical polar	Cylindrical
$F_x = m\ddot{x}$	$F_r = m\left(\ddot{r} - r\dot{\theta}^2\right)$	$F_\rho = m\left(\ddot{\rho} - \rho\dot{\phi}^2\right)$
$F_y = m\ddot{y}$	$F_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$	$F_\phi = m\left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)$
$F_z = m\ddot{z}$		$F_z = m\ddot{z}$

EXAMPLE 1.17

Example 1.17 If the position of a particle is $\vec{r} = \hat{i}\cos\omega t + \hat{j}\sin\omega t$, show that its velocity is perpendicular to $\vec{r}$ and $\vec{r}\times\vec{v}$ is a constant vector.

Solution

Here $\quad \vec{r} = \hat{i}\cos\omega t + \hat{j}\sin\omega t$

So, the velocity of the particle is

$$\vec{v} = \frac{d\vec{r}}{dt} = -\hat{i}\,\omega\sin\omega t + \hat{j}\omega\cos\omega t$$

Thus, $\quad \vec{r}.\vec{v} = \left(\hat{i}\cos\omega t + \hat{j}\sin\omega t\right).\left(-\hat{i}\,\omega\sin\omega t + \hat{j}\omega\cos\omega t\right) = 0$

So, the velocity of the particle is perpendicular to $\vec{r}$.

Again, $\quad \vec{r}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix}$

$$= \hat{k}\omega$$

Thus, $\vec{r}\times\vec{v}$ is a constant vector.

EXAMPLE 1.18

Example 1.18 For a moving particle in a plane, the polar coordinates are given by $r = a\sin\omega_1 t$ and $\theta = \omega_2 t$. Find an expression for the polar components of the velocity and acceleration of the particle.

Solution

Here, the radial and cross radial components of velocities are

$$v_r = \dot{r} = a\omega_1\cos\omega_1 t$$

and $\quad v_\theta = r\dot{\theta} = r\omega_2$

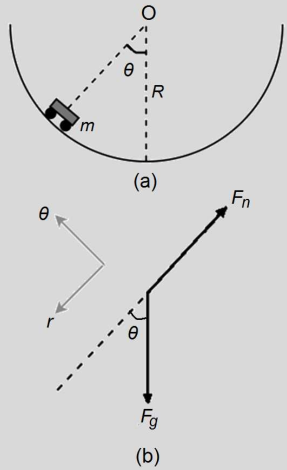
Similarly, the radial and cross radial components of accelerations are

$$f_r = \ddot{r} - r\dot{\theta}^2 = -r\omega_1^{\,2} - r\omega_2^{\,2} = -r\left(\omega_1^{\,2} + \omega_2^{\,2}\right)$$

and $\quad f_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 2\dot{r}\dot{\theta} = 2a\omega_1\omega_2\cos\omega_1 t \qquad$ [As $\ddot{\theta} = 0$]

Example 1.19 Assume that a skateboard of mass m rolls without friction on a semicircular half pipe of radius R. Neglecting the resistance determine an expression for $\theta(t)$ considering that the skateboard starts from rest at an angle θ_0.

Solution



When Newton's laws are considered in polar coordinates we have,

$$F_r = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

and $$F_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

If implemented these to the free body system in figure we have,

$$F_g \cos\theta - F_n = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

[for radial direction]

and $$-F_g \sin\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

[for cross-radial direction]

or, $$-mg\sin\theta = mr\ddot{\theta} \qquad \text{[as } r = R \text{ is constant]}$$

$$\therefore \quad \ddot{\theta} = -\frac{g}{R}\sin\theta$$

Let us make one assumption about the motion, that θ is always small. Then $\sin\theta \approx \theta$.

$$\therefore \quad \ddot{\theta} = -\frac{g}{R}\theta$$

The general solution of the above equation is of the form,

$$\theta(t) = A\sin\sqrt{\frac{g}{R}}t + B\sin\sqrt{\frac{g}{R}}t$$

Now, $$\theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = 0$$

Using these conditions in above equation we get $B = \theta_0$ and $A = 0$

Therefore, $$\theta(t) = \theta_0 \sin\sqrt{\frac{g}{R}}t.$$

EXAMPLE 1.19

UNIT SUMMARY

- ***Vector fundamentals***

Components of a vector

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\vec{A} = \left|\vec{A}\right|\hat{n}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Vector addition

Commutative law: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

Associative law: $\left(\vec{A} + \vec{B}\right) + \vec{C} = \vec{A} + \left(\vec{B} + \vec{C}\right)$

$$\vec{A} - \vec{B} = \vec{A} + \left(-\vec{B}\right)$$

$$m\left(\vec{A} + \vec{B}\right) = m\vec{A} + m\vec{B}$$

Vector product (multiplication)

$$\vec{A}.\vec{B} = AB\ \cos\theta$$

$$\vec{A} \times \vec{B} = AB\sin\theta\hat{n}$$

The del operator

Cartesian coordinates: $\vec{\nabla} = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$

Cylindrical coordinates: $\vec{\nabla} = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \varphi}\hat{a}_\varphi + \frac{\partial}{\partial z}\hat{a}_z$

Spherical polar coordinates: $\vec{\nabla} = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\hat{a}_\varphi$

Gradient of a scalar function

$$\text{grad}\, V = \frac{\partial V}{\partial n}\hat{a}_n = \vec{\nabla}V$$

Divergence of a vector field

$$\text{div}\vec{A} = \vec{\nabla}.\vec{A} = \lim_{\Delta V \to 0}\frac{\oint_s \vec{A}.d\vec{s}}{\Delta V}$$

Curl of a vector field

$$\text{curl}\vec{A} = \vec{\nabla} \times \vec{A} = \lim_{\Delta S \to 0}\frac{\hat{a}_n}{\Delta S}\left[\oint_C \vec{A}.d\vec{l}\right]_{\max}$$

Laplacian of a scalar field

$$\nabla^2\phi = \vec{\nabla}.\vec{\nabla}\phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

- ***Coordinate systems***

General formulation

$$\hat{a}_u \times \hat{a}_v = \hat{a}_w, \hat{a}_v \times \hat{a}_w = \hat{a}_u, \hat{a}_w \times \hat{a}_u = \hat{a}_v$$

$$\hat{a}_u.\hat{a}_v = \hat{a}_v.\hat{a}_w = \hat{a}_w.\hat{a}_u = 0, \hat{a}_u.\hat{a}_u = \hat{a}_v.\hat{a}_v = \hat{a}_w.\hat{a}_w = 1$$

$$\vec{A} = A_u\hat{a}_u + A_v\hat{a}_v + A_w\hat{a}_w$$

Cartesian coordinate system

$$\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$$

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z, \hat{a}_y \times \hat{a}_z = \hat{a}_x, \hat{a}_z \times \hat{a}_x = \hat{a}_y$$

$$\hat{a}_x.\hat{a}_y = \hat{a}_y.\hat{a}_z = \hat{a}_z.\hat{a}_x = 0$$

$$\hat{a}_x.\hat{a}_x = \hat{a}_y.\hat{a}_y = \hat{a}_z.\hat{a}_z = 1$$

$$d\vec{l} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz$$

$$d\vec{S}_x = dydz\hat{a}_x \quad d\vec{S}_y = dxdz\hat{a}_y \quad d\vec{S}_z = dxdy\hat{a}_z$$

$$dV = dxdydz$$

Cylindrical coordinate System

$$\vec{A} = \hat{a}_\rho A_\rho + \hat{a}_\phi A_\phi + \hat{a}_z A_z$$

$$\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z, \hat{a}_\phi \times \hat{a}_z = \hat{a}_\rho, \hat{a}_z \times \hat{a}_\rho = \hat{a}_\phi$$

$$\hat{a}_\rho.\hat{a}_\phi = \hat{a}_\phi.\hat{a}_z = \hat{a}_z.\hat{a}_\rho = 0$$

$$\hat{a}_\rho.\hat{a}_\rho = \hat{a}_\phi.\hat{a}_\phi = \hat{a}_z.\hat{a}_z = 1$$

$$d\vec{l} = \hat{a}_\rho d\rho + \rho\hat{a}_\phi d\phi + \hat{a}_z dz$$

$$d\vec{S}_\rho = \rho d\phi dz\hat{a}_\rho \quad d\vec{S}_\phi = d\rho dz\hat{a}_\phi \quad d\vec{S}_z = \rho d\rho d\phi\hat{a}_z$$

$$dV = \rho d\rho d\phi dz$$

$$x = \rho\cos\phi, y = \rho\sin\phi \text{ and z = z}$$

$$\rho = \sqrt{x^2 + y^2}, \phi = \tan^{-1}\frac{y}{x} \text{ and z = z}$$

Spherical polar coordinates system

$$\vec{A} = \hat{a}_r A_r + \hat{a}_\theta A_\theta + \hat{a}_\phi A_\phi$$

$$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi, \hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r, \hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta$$

$$d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r d\theta + \hat{a}_\phi r\sin\theta d\phi$$

$$d\vec{S}_r = r^2\sin\theta d\theta d\phi\hat{a}_r \quad d\vec{S}_\theta = r\sin\theta dr d\phi\hat{a}_\theta \quad d\vec{S}_\phi = r dr d\theta\hat{a}_\phi$$

$$dV = r^2\sin\theta dr d\theta d\phi$$

$$z = x^2 + y^2$$

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi \text{ and } z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2},\ \theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} \text{ and } \phi = \tan^{-1}\frac{y}{x}$$

- ***Transformation properties of vectors and scalars***

  Transformation of vector components

$$\begin{pmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{pmatrix} = \lambda \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

  Transformation of the scalar product

$$\left(\vec{A}.\vec{B}\right)' = A_{x'}.B_{x'} = \lambda_{xy}A_y\lambda_{xz}B_k = \delta_{jk}A_yB_k = A_yB_y = \vec{A}.\vec{B}$$

  Transformation of the vector product

$$\vec{C}' = \begin{vmatrix} \hat{e}_{x'} & \hat{e}_{y'} & \hat{e}_{z'} \\ A_{x'} & A_{y'} & A_{z'} \\ B_{x'} & B_{y'} & B_{z'} \end{vmatrix} = (-1)^3 \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = -\vec{C}$$

- ***Fundamental forces of nature***

  Gravitational force: 6×10^{-39} times weaker than the strong force.

  Strong force: It acts over a range of about 10^{-15} m.

  Weak force: It has the shortest range of 10^{-18} m.

  Electromagnetic force: Strength is 1/137 times of strong force

- ***Newton's laws of motion***

  First law: Everybody continues its state of rest or of uniform motion.

  Second law: $\vec{F} = m\vec{a}$

  Third law: To every action there is an equal and opposite reaction.

- ***Momentum***

$$\vec{p} = m\vec{v}$$

- ***Principle of conservation of linear momentum***

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

  Recoil of a gun $V = -\dfrac{mv}{M}$

- ***Motion of two bodies connected by inextensible weightless string***

$$f = \frac{F}{m_1 + m_2} \qquad T = m_1 f = \frac{m_1 F}{m_1 + m_2}$$

- ***Vertical motion of two bodies connected by inextensible string***

$$F = (m_1 + m_2)(g + f)$$

$$T = m_2(g + f)$$

- ***Conservation of mechanical energy***

  The sum of KE and PE of a body is its total mechanical energy.

- ***Friction***

  Friction is a force resistive to the direction of motion.

  $$\mu = \frac{F}{N}$$

- ***Laws of friction***

  First law: The force of friction is directly proportional to the applied load.
  Second law: Friction force is independent of the apparent area of contact.
  Third law: Kinetic friction is independent of the sliding velocity.

- ***Angle of friction***

  $$\tan\theta = \frac{F}{N} = \mu_s$$

- ***Limitations of Newton's laws***

  i) Newton's laws are only valid in the Cartesian-like inertial frames.
  ii) It requires complete specification of all the forces acting on the body.

- ***Inertial reference frame***

  A frame at rest or moving with uniform speed. Newton's laws are valid in this reference frame.

- ***Cartesian coordinates***

  $$F_x = m\ddot{x} \qquad F_y = m\ddot{y} \qquad F_z = m\ddot{z}$$

- ***Polar coordinate system***

  $$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\frac{y}{x}$$

- ***Polar coordinates in a rotating frame***

  $$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \qquad \vec{f} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

EXERCISES

Multiple Choice Questions

1.1 The angle between $\hat{i}$ and $\hat{i} + 2\hat{j}$ is?

(a) $\cos^{-1}\frac{2}{5}$ (b) $\cos^{-1}\frac{2}{\sqrt{5}}$ (c) $\cos^{-1}\frac{\sqrt{2}}{5}$ (d) $\cos^{-1}\sqrt{\frac{2}{5}}$

1.2 If a vector field is given by $\vec{F} = \frac{\sinh(r)}{1+2r^3}$ then $\vec{\nabla} \times \vec{F}$

(a) $3\vec{r}$ (b) zero (c) $\left[\frac{\cosh(r)}{1+2r^3} - 3r\right]\hat{\theta}$ (d) $\left[\frac{\cosh(r)}{1+2r^3} - 3r\right]\vec{F}$

1.3 The angle between the vector $\hat{i} + \hat{j}$ and $\hat{i} - \hat{j}$ is

(a) 90° (b) 60° (c) 30° (d) 0°

1.4 Unit vector in the direction of $\vec{A}$ is

(a) $\frac{\vec{A}}{A}$ (b) $\vec{A}A$ (c) $\frac{A}{\vec{A}}$ (d) $\frac{1}{\vec{A}A}$

1.5 The value of $\vec{\nabla}.\vec{r}$ is

(a) 0 (b) 1 (c) 3 (d) 2

1.6 The value of $\vec{\nabla}\times\vec{r}$ is equal to

(a) 1 (b) 0 (c) -1 (d) 2

1.7 When the magnitude of $\vec{A}$ is constant which one of the following is true?

(a) $\frac{d\vec{A}}{dt}=0$ (b) $\vec{A}.\frac{d\vec{A}}{dt}=0$ (c) $\vec{A}\times\frac{d\vec{A}}{dt}=0$ (d) $\left|\frac{d\vec{A}}{dt}\right|=0$

1.8 The volume element in spherical polar coordinate is

(a) $r\sin\theta dr d\theta d\phi$ (b) $r^2\sin\theta dr d\theta d\phi$ (c) $\sin\theta dr d\theta d\phi$ (d) $r^2\sin^2\theta dr d\theta d\phi$

1.9 A ball of mass 5 kg strikes a wall with a speed of 10 m/s and rebounds with the same speed. The change of the momentum of ball is

(a) zero (b) 50 kg-m/s (c) 100 kg-m/s (d) none of these

1.10 A truck and a motor car are moving with same momentum. Same force acts separately against them to make them stop. Which one will cross lowest distance?

(a) truck (b) motor car

(c) both (d) they will not stop at all

1.11 Newton's laws of motion is applicable for

(a) frame of rest (b) inertial frame (c) non-inertial frame (d) dynamical frame

1.12 If a bullet is fired into a window made of glass, then place of target makes a hole because of

(a) inertia of rest (b) inertia of motion

(c) change of momentum (d) none of these

1.13 A body of mass m_1 collides elastically with a rest body of mass m_2, calculate the fraction of KE that is transformed into the body of mass m_2 from m_1

(a) m_1/m_2 (b) m_2/m_1 (c) $\frac{2m_1m_2}{(m_1+m_2)^2}$ (d) $\frac{4m_1m_2}{(m_1+m_2)^2}$

1.14 If the normal reaction is doubled, then the change of coefficient of friction will be

(a) doubled (b) halved (c) unchanged (d) none of these

1.15 Conservation of linear momentum may be obtained from

(a) Newton's 1st law of motion (b) Newton's 2nd law of motion

(c) Newton's 3rd law of motion (d) law of conservation of mass

1.16 A boy in a train which in moving with constant velocity threw a ball vertically upward in that compartment, then the ball

(a) will reach the boy's hand (b) will reach in front of the body
(c) will fall on behind him (d) will stay in air

1.17 The working principle of jet is
(a) law of conservation of mass (b) law of conservation of energy
(c) law of conservation of linear momentum (d) law of conservation of angular momentum

1.18 If m is the mass of a body and E its kinetic energy increases
(a) $m\sqrt{E}$ (b) $\sqrt{mE}$ (c) $2\sqrt{mE}$ (d) $\sqrt{2mE}$

1.19 When mass and speed are doubled, the kinetic energy increases
(a) 4 times (b) 2 times (c) 16 times (d) 8 times

1.20 If the kinetic energy of a given body is doubled, its momentum will
(a) become 1/2 (b) remain unchanged (c) become $\sqrt{2}$ times (d) doubled

1.21 Two bodies of masses m_1 and m_2 are with equal kinetic energy. The ratio of their linear momentum p_1, p_2 is
(a) $m_1 : m_2$ (b) $\sqrt{m_1} : \sqrt{m_2}$ (c) $m_2 : m_1$ (d) $m_1^2 : m_2^2$

1.22 A car moving with uniform velocity over a rough horizontal surface. According to Newton's first law of motion
(a) the kinetic energy of the car will increase (b) the engine will apply no force
(c) acceleration be produced
(d) the force produced by engine will balance the frictional force

1.23 Rocket propulsion is based on the principle of conservation of
(a) linear momentum (b) mass (c) kinetic energy (d) none of these

1.24 A gun of mass 10 kg fires a bullet of mass 40 gm with a speed of 400 m/s. The recoil velocity of the gun is
(a) 1.6 m/s (b) 0.16 m/s (c) 160 m/s (d) 16 m/s

1.25 A frame moving relative to an initial frame with uniform velocity
(a) must be inertial (b) must be non-inertial
(c) cannot be inertial (d) may or may not be inertial

Answers of Multiple Choice Questions

1.1 (b), 1.2 (b), 1.3 (a), 1. 4 (a), 1.5 (c), 1.6 (b), 1.7 (b), 1.8 (b), 1.9 (c), 1.10 (a), 1.11 (b), 1.12 (a), 1.13 (d), 1.14 (c), 1.15 (c), 1.16 (a), 1.17 (c), 1.18 (d), 1.19 (d), 1.20 (c), 1.21 (b), 1.22 (c), 1.23 (b), 1.24 (a), 1.25 (a)

Short and Long Answer Type Questions

Category I

1.1 Explain scalar and vector fields with example.

1.2 Define curl of a vector field.

1.3 Show that the total energy is conserved for the free fall of a body.

1.4 Explain the law of conservation of energy.

1.5 Establish Newton's third law from second law. Use the principle of conservation of linear momentum.

1.6 Find the relation between the Cartesian coordinates (x, y, z) of a point in terms of its cylindrical counterparts.

1.7 Explain the principle of conservation of linear momentum.

1.8 Establish the principle of conservation of linear momentum from Newton's 2^{nd} law and 3^{rd} law.

1.9 A particle is acted upon by two constant forces $\hat{i}+4\hat{j}-3\hat{k}$ and $3\hat{i}+\hat{j}-\hat{k}$, so that it is displaced from the point $\hat{i}+3\hat{j}+3\hat{k}$ to the point $4\hat{i}+5\hat{j}+\hat{k}$. Find the total work done.

1.10 Prove that for the motion of a particle in a plane

$$\vec{v}=\dot{r}\hat{e}_r+r\dot{\theta}\hat{e}_\theta \text{ and } \vec{f}=\left(\ddot{r}-r\dot{\theta}^2\right)\hat{e}_r+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

where the symbols have their usual significance.

1.11 Show that the velocity of a particle moving in a three dimensional space is

$$\vec{v}=\dot{r}\hat{e}_r+r\dot{\theta}\hat{e}_\theta+r\sin\theta\,\dot{\phi}\,\hat{e}_\phi$$ where the symbols have their usual meaning.

Give the physical significance of each term.

1.12 Write down the Cartesian coordinates (x, y, z) in terms of spherical polar coordinates (r, θ, ϕ).

1.13 State the principle of conservation of linear momentum for a single particle.

Category II

1.14 a) A heavy body and a lighter body possess equal momentum. Which one has greater kinetic energy?

b) Show that lane of conservation of energy and illustrate it by discussing the energy changes that occur when a body is allowed to fall freely from a height h.

c) A man moves up in a lift carrying a suitcase in his hand. Explain if any work is the man on suitcase.

1.15 a) Explain the conservation of energy. Prove that in a freely falling body conservation of energy is followed.

b) What is inclined plane? How you can measure energy of a body falling through inclined plane.

1.16 Use the expression for gradient in the spherical coordinate system to find the normal to the surface $r\sin\theta = 1$.

1.17 Define unit vectors $\hat{r}$ and $\hat{\theta}$ in planar motion in terms of their Cartesian counterparts $\hat{i}$ and $\hat{j}$, hence show that

$$\frac{d\hat{r}}{d\theta}=\hat{\theta} \text{ and } \frac{d\hat{\theta}}{d\theta}=-\hat{r}.$$

Numerical Problems

1.1 A body is pulled with constant velocity over a rough horizontal surface through 5 m by applying a force of 20 N at an angle of 30^0 with the horizontal. Find the work down by the force the frictional force the frictional force and work down by a frictional force.

1.2 The momentum of two bodies of masses M_1 and M_2 are same. What will be the ratio of their kinetic energy?

1.3 A bullet of mass 0.03 kg moving at a speed of 500 ms^{-1} penetrates 12 cm into a fixed block of wood. Calculate the average force exerted by the wood on the bullet.

1.4 A bullet moving with a velocity of 100 m/s can penetrate 1 m into a target. If an another bullet of same type penetrates into a same type target of thickness 0.5 m then calculate the velocity after penetration.

1.5 How much work is to be done to drag a body of mass 200 kg through a distance of 80 m on a leveled road, the co-efficient of friction between the road and the body being 0.25?
[*Ans*: 39200 J]

1.6 A bullet of mass 20 gm moving with a velocity of 200 ms^{-1} is stopped within 40 cm of the target. Calculate the force offered by the target on the bullet.

1.7 A train weighing 106 kg is moving up an inclined plane rising 1 m, at a uniform speed of 72 kmh^{-1}. If the frictional resistance amounts to 0.4 kg per metric ton, find the power of the engine.
[*Ans*: 1058.4 kW]

1.8 A 8 gm bullet is shot from a 5 kg gun with a speed of 400 m/s. Find the recoil velocity of the gun. [*Ans*: 64 cm/s]

1.9 A 50 kg crate is being pushed on a horizontal floor at constant velocity. Given that the coefficient of kinetic friction between crate and floor is $\mu_k = 0.1$. What is the required push force F? [*Ans*: 49 N]

1.10 Show that the plane polar coordinates are orthogonal.

1.11 Apoint moving in a plane has coordinates (3, 4) and has components of speed (5, 8) at some instant of time. Find the components of speed in terms of (r, θ) along the directions ($\hat{r}, \hat{\theta}$).

1.12 The polar coordinates of a point is $(8, 30^\circ, 45^\circ)$. Find the Cartesian coordinates of the same point. [*Ans*: $2\sqrt{3}, 2\sqrt{2}, 4\sqrt{3}$]

1.13 Assuming the expressions for the radial and the cross radial components of accelerations in two dimensional polar coordinates, show that a body in circular motion with a constant angular velocity will experience an acceleration v^2/r towards the centre only.

PRACTICAL

Experiments on an Air-Track

Aim

Investigate motion of an accelerated glider on an air track to see the dependence of acceleration on the mass of the glider and the tilt angle

Apparatus

Air track, Gliders, Air pulley, Slotted weights, Proper magnetic recording tape, Spark recording tape, Riser blocks

Theory

When set in motion on a level track a glider moves with very little friction and therefore effectively moves with nearly constant velocity. If a force is applied on the glider, it can be accelerated.

This can be achieved by attaching proper length of mylar magnetic recording tape passing over an air pulley to the glider at one end of the track, and attaching small slotted weights to the other (free) end. This weight must be small compared to the weight of the glider so that the velocity of the glider can be kept very small at all times.

The tension in the mylar tape is

$$T = m(g - a) \qquad (i)$$

where m is the hanging mass and a is the acceleration of the glider.

So, the force exerted by the tape on glider can be taken as the tension in the tape, *i.e.*,

$$T = Ma = m(g - a) \qquad (ii)$$

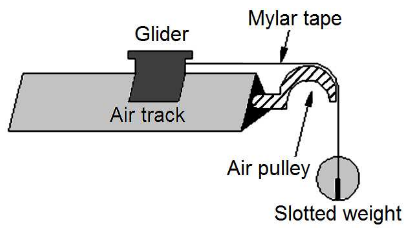


Fig. (i)

Thus, the glider's acceleration is

$$a = \frac{mg}{M + m} \qquad (iii)$$

where M is the mass of the glider. The relation between a and m is therefore *not* strictly linear.

Procedure

1. First investigate the relation between the glider acceleration and force applied to observe whether it is in agreement with Newton's second law ($F = ma$). The applied force F is very nearly equal to the weight of the hanging mass ($F \sim W$).

2. Take data for a number of different glider masses, m. For each mass m, use different weights, W, and measure the acceleration, a. Take reading for at least 5 different glider masses, and 5 different weights.

3. For each case using $W = ma$ calculate the expected acceleration theoretically. Compare this with the experimentally measured acceleration for each case and note the discrepancy between theory and experiment.

4. Plot a graph between W and a for each glider. This graph will be a straight line with a slope $1/m$. For every glider there will be different slopes. These can be plotted on a single graph paper. Calculate the slope of each of the lines on the graphs.

Now, on another graph paper, plot these slopes against the corresponding mass m. This can be used to explain the agreement between theory and data. Alternatively you can plot all of the data for a single W and draw a graph between a and $1/m$.

If graph for all W are plotted on a single paper, there will be different straight line for each W.

5. Now, place riser blocks of varying thickness below the leg at one side of the track for making a small tilt angle. The glider will accelerate on the tilted track as the force of gravity on the glider will have a component parallel to the track.

No other accelerating force will be used in this part of the experiment. Measure the glider acceleration for different tilt angles for each value of the glider mass.

6. Record four tracks of data on each tape. Conserve the data tape keeping four different records on each of the tape. You have to do it slightly bending the spark wire on glider for locating the same at any one of the four parallel tracks in the direction of the waxed tape.

Thus you will need only four lengths of tape for this part of the experiment.

KNOW MORE

At the fundamental level, most of the phenomena in the world are based on physics and many parts of physics are based on mechanics. This permits us to analyze the operation of majority of familiar phenomena around us like planets, stars and galaxies. In fact, all of the physics as well as different branches of engineering are mapped into the procedures and diagnostics where the mechanics are to widely use and in the data tables.

Interestingly, during the seventeenth century, classical mechanics was developed with Sir Isaac Newton's laws of motion in '*Philosophiae Naturali Principia Mathematica*'. He was the first to unify the three laws of motion, *viz*., the law of inertia, his second law and the law of action and reaction. He proved successfully that these laws govern both earthly and celestial objects.

Activity

Classical physics is largely concentrated on the study of motion, pulleys, projectiles and the planets. Primarily it deals with the movement of heavy objects through space at comparatively low speeds. Basically, classical physics gives priority to the mechanics when the motion of an object comes into play in response to a force. Due to this, classical physics is frequently referred to simply as mechanics or kinematics. Classical mechanics gives extremely accurate results when considered large objects which are not extremely massive and also the speed not approaching to the speed of light. When the objects have about the size of an atom diameter, it required to implement other major sub-field of mechanics related to quantum mechanics.

Sir Isaac Newton

To explain velocities not small compared to the speed of light, special relativity is essential. On the other hand, when objects become extremely massive, general relativity becomes applicable.

Interesting facts

One day at his time, Sir Isaac Newton saw an apple to fall on the ground and he figured out that the same force which makes the apple to fall also can govern the motion of the Moon and the planets.

In 1687 he published his three laws of motion in the "Principia Mathematica Philosophiae Naturalis" which explains how the concepts of force and motion work.

Sir Isaac Newton wrote more about the Bible and on the religion rather than about Mathematics, Physics or Astronomy. Mainly he studied the Bible to extract the inherent scientific information in the Bible. In 1704, Newton wrote a manuscript, which contained different scientific notes based on the Holy Bible.

Analogy

Engineering mechanics is mainly associated with the application of mechanics for solving problems which are involved in common engineering elements. It is important to note that the goal of the Engineering Mechanics is to emphasize problems in mechanics as mostly applied to plausibly real-world scenarios.

Within the field of practical sciences, applied mechanics is most useful to formulate new ideas and theories, discovering and interpreting phenomena as well as developing experimental and computational tools.

When we examine the field of the application of the natural sciences we find that mechanics is said to be complemented by thermodynamics, the study of heat and more generally to the areas of energy and electro-mechanics. Though the classical mechanics is largely compatible with other theories of *classical physics* like classical electrodynamics and thermodynamics, some serious problems were noted in the late 19th century. These problems could only be resolved by more modern physics.

When combined with the classical thermodynamics it was found that the classical mechanics leads to the Gibbs paradox where entropy is not a well-defined quantity. The attempt to resolve these types of problems led to the development of quantum mechanics. In a similar way, the variety of behaviors of classical electromagnetism and classical mechanics under velocity transformations led to the development of theory of relativity.

History

The ancient Greek philosophers, Aristotle in particular, were among the first to propose that abstract principles govern nature. Aristotle argued, in *On the Heavens*, that terrestrial bodies rise or fall to their natural place and stated as a law the correct approximation that an object's speed of fall is proportional to its weight and inversely proportional to the density of the fluid it is falling through.

Timelines

1687: *Isaac Newton* published *Philosophiae Naturalis Principia Mathematica*, where he discussed laws of motion and laws of universal gravitation.

1691: *Johann Bernoulli* showed that a freely suspended chain from two points will form a catenary.

1691: *James Bernoulli* showed that the catenary curve has the lowest center of gravity of any chain hung from two fixed points.

1694: *Leibniz* used the word "coordinate" in its modern usage

1747: *Maupertuis* considered the Principle of Least Action and defined as a variable having the same dimensions as length times momentum or energy times time.

1776: *John Smeaton* published a paper on experiments associated with power, work, momentum, kinetic energy supporting the conservation of energy.

1789: *Antoine Lavoisier* stated the law of conservation of mass.

1813: *Peter Ewart* supports the idea of the conservation of energy through his paper "On the measure of moving force".

1834: *Jacobi* *n*-fold integrals and volumes of *n*-dimentional spheres.

1834: *Hamiltonian* formulated the principle that the path taken by a physical system minimizes the integral

-35 of the difference between kinetic and potential energy.

1841: *Julius Robert von Mayer* wrote a paper on the conservation of energy but his lack of academic training leads to its rejection.

1847: *Hermann von Helmholtz* formally stated the law of conservation of energy.

1881: *Gibbs* modifies vector analysis.

1915: *Emmy Noether* proved Noether's theorem, from which conservation laws were deduced.

1983: *Mordehai Milgrom* proposed Modified Newtonian dynamics.

Applications (Real Life / Industrial)

For most of the practical applications we are interested in the position or the velocity of the particle as a function of time. However, Newton's laws will only tell us its acceleration. So, for all practical purposes scientists established equations that relates the position, velocity and acceleration.

The mathematical formulation and proof of the unified mechanics theory is based on the unification of Newton's laws and the laws of thermodynamics. It presents formulations and experimental verifications of the theory for thermal, mechanical, electrical, corrosion, chemical and fatigue loads and explains the original universal laws of motion proposed by Isaac Newton in 1687.

It provides many concrete examples, such as how Newton's second law, $F = ma$, gives the initial acceleration of a soccer ball kicked by a player but does not tell us how and when the ball would come to a stop.

Unified mechanics theory with extensive experimental validation provides many finite element implementation using real world examples. It draws the connections to the thermodynamics of degradation in solids from mathematical and micro-structural perspective.

Case Study (Environmental / Sustainability / Social / Ethical Issues)

The development in applied mechanics exhibits wide application in many fields of study. Some of the special areas which put the subject into practice are Mechanical Engineering, Materials Science, Civil Engineering, Aerospace Engineering, Bio-engineering and many more. Applied mechanics thus bridges the gap between the physical theory and its various applications in technology.

The phase space in classical mechanics permits a natural description as a symplectic manifold. It is indeed a cotangent bundle in most of the cases of physical interest and symplectic topology. This can be thought of as the study of global issues of Hamiltonian mechanics. Since the 1980s this has become a fertile area of mathematics research.

Inquisitiveness and Curiosity Topics

Modern applied mechanics is based on Isaac Newton's laws of motion while the modern practice of their applications can be traced back to Stephen Timoshenko, who is considered as the father of modern engineering mechanics.

At the end of the 20th century, classical mechanics in physics was no longer an independent theory. With classical electromagnetism, it has become embedded in quantum field theory or relativistic quantum mechanics. It now defines clearly the non-relativistic and non-quantum mechanical limit for massive particles. For mathematicians, classical mechanics has got a source of inspiration.

Stepan Prokopovych Tymoshenko, Father of modern engineering

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2 Conservation Principles

UNIT SPECIFICS

This unit is focussed on the following main aspects:

- Potential energy function and relationship with force;
- Equipotential surfaces and meaning of gradient;
- Conservative and non-conservative forces, curl of a force field;
- Central forces;
- Conservation of angular momentum;
- Energy equation and energy diagrams;
- Elliptical, parabolic and hyperbolic orbits;
- Kepler problem, satellite manoeuvres.

The applications of the topics are discussed for generating further curiosity and creativity as well as improving problem solving capacity.

Besides giving a large number of multiple choice questions as well as questions of short and long answer types marked in two categories following lower and higher order of Bloom's taxonomy, assignments through a number of numerical problems, a list of references and suggested readings are given in the unit so that one can go through them for practice. It is important to note that for getting more information on various topics of interest some QR codes have been provided in different sections which can be scanned for relevant supportive knowledge.

After the end of the unit, based on the content, there is a "Know More" section. This section has been carefully designed so that the supplementary information provided in this part becomes beneficial for the users of the book. This section mainly highlights the initial activity, examples of some interesting facts, analogy, history of the development of the subject focusing the salient observations and finding, timelines starting from the development of the concerned topics up to the recent time, applications of the subject matter for our day-to-day real life or/and industrial applications on variety of aspects, case study related to environmental, sustainability, social and ethical issues whichever applicable, and finally inquisitiveness and curiosity topics of the unit.

Rationale

This unit on Conservation Principles will help our students to get a clear idea about the potential energy function, equi-potential surfaces and meaning of gradient as well as Conservative and non-conservative forces. It establishes some theoretical relations showing curl of a force field, central forces, conservation of angular momentum and energy equation and energy diagrams. In addition to these theoretical approaches elliptical, parabolic and hyperbolic orbits, as well as Kepler problem and various applications including Satellite manoeuvres are illustrated

The principle of energy conservation states that energy can be neither created nor destroyed. Only it may transform from one type to another. Some of the common forms of energy include thermal, mechanical, electrical, chemical, kinetic and potential. It is said that the sum of all types of energy is constant. The laws of conservation of energy, momentum and angular momentum all can be derived from the classical mechanics. From the conservation principles we get that some quantity, quality or aspect remains constant through change.

PRE-REQUISITES

Mathematics: Vector Calculus (Class XII)

Physics: Mechanics, General Properties of Matter (Class XII)

UNIT OUTCOMES

List of outcomes of this unit is as follows:

U2-O1: Describe conservative and non-conservative forces

U2-O2: Define central force and central orbit

U2-O3: Explain conservation of momentum, energy and angular momentum

U2-O4: Explain the reduction of two bodies Kepler problem to a one body problem

U2-O5: Apply energy and momentum conservation laws to central force motion problems

U2-O6: Analyze energy diagrams in different energy level solutions

Unit-2 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (*1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation*)					
	CO-1	CO-2	CO-3	CO-4	CO-5	CO-6
U2-O1	1	2	2	3	3	3
U2-O2	-	2	1	3	2	-
U2-O3	2	3	2	2	2	1
U2-O4	-	1	1	3	-	-
U2-O5	-	2	1	3	1	-
U2-O6	-	-	1	3	-	-

2.1 INTRODUCTION

We are familiar with a number of conservation laws in physics. These laws include i) conservation of energy, ii) conservation of linear momentum, iii) conservation of angular momentum etc. These laws are very useful to solve different problems in physics which are the general consequence of specific underlying symmetry of the Universe.

Johannes Kepler was the first to formulate the laws of planetary motion which can be used to solve the planetary equation of motion. Such problems are known as the Kepler problems. As for a three body problem, like the motion of the moon which gravitationally interacts with the Earth and the Sun, it is difficult to solve by exact method. Further investigations on the three body problem, were put forward by Leonhard Euler to arrive at the question that can provide the feeble contribution of gravitational interaction from the planets making the planetary system unstable.

2.2 CONSERVATIVE AND NON-CONSERVATIVE FORCES

2.2.1 Conservative forces

If in a force field the work done for the displacement of a particle from first point to the second point does not depend on the path along which the particle is displaced but only depends on the initial and final position of the particle, then such force is known as the *conservative force*; *e.g.*, gravitational force, electrostatic force *etc*.

The amount of work done in the conservative force field is path independent and the total mechanical energy for a conservative force field remains always constant.

The amount of work done in a non-conservative force field is path dependent.

In Fig. 2.1 the particle is taken from A to B along two different paths AMB and ANB. If in each time the amount of work done by the force is same, then this force is conservative. Hence for a conservative force

$$\int_{\substack{A \\ (\text{Path } AMB)}}^{B} \vec{F}.d\vec{r} = \int_{\substack{A \\ (\text{Path } ANB)}}^{B} \vec{F}.d\vec{r}$$

Here, AMB and ANB are any two arbitrary paths joining the points A and B. The region of the conservative force is called *conservative force field*. In such a force field work done over a closed path is zero *i.e.*,

$$\oint_C \vec{F}.d\vec{r} = 0$$

Now, using Stokes' theorem we have

$$\oint_C \vec{F}.d\vec{r} = \int_S \vec{\nabla} \times \vec{F}.d\vec{S}$$

where, S is a surface bounded by C. Hence for a conservative force field we should have

$$\int_S \vec{\nabla} \times \vec{F}.d\vec{S} = 0$$

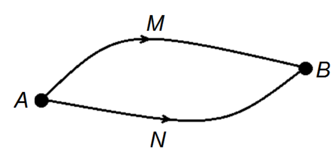

Fig. 2.1: Path integral of force in conservative force field

As it is applicable to any closed path, so for a conservative force field,

$$\vec{\nabla} \times \vec{F} = 0 \tag{2.1}$$

Now we define a scalar quantity V which is a function of the co-ordinates in the force field. Since $\vec{\nabla} \times \vec{F} = 0$ the force $\vec{F}$ of Eq. (2.1) may be expressed as gradient of some a scalar function. Hence we put $\vec{F}$ as,

$$\vec{F} = -\vec{\nabla} V \tag{2.2}$$

Potential function V defined by Eq. (2.2) is meaningful only for a conservative field and can always be defined in such a conservative field. The potential function V is a scalar quantity. It can easily provide the components of force $\vec{F}$ at any point as below,

$$\vec{F} = \left(\hat{i}F_x + \hat{j}F_y + \hat{k}F_z\right) = -\left(\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right)$$

So, $$F_x = -\frac{\partial V}{\partial x} \tag{2.3 (a)}$$

$$F_y = -\frac{\partial V}{\partial y} \tag{2.3 (b)}$$

and, $$F_z = -\frac{\partial V}{\partial z} \tag{2.3 (c)}$$

In spherical polar form, however, we have

$$\vec{F} = \left(\hat{r}F_r + \hat{\theta}F_\theta + \hat{\phi}F_\phi\right) = -\left(\hat{r}\frac{\partial V}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial V}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\right)$$

So, $$F_r = -\frac{\partial V}{\partial r} \tag{2.4 (a)}$$

$$F_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} \tag{2.4 (b)}$$

and, $$F_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} \tag{2.4 (c)}$$

Like all vectors, the gradient has both the magnitude and direction. To have an idea about gradient properly let us take the dot product of say $\vec{\nabla} V$ with $d\vec{r}$ such that

$$\vec{\nabla} V . d\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) V . \left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)$$

$$= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = dV \tag{2.5 (a)}$$

where $V = V(x, y, z)$ represents a scalar function of positional coordinates (x, y, z) of a particle. Now, we can write,

$$\vec{\nabla}V.d\vec{r} = \left|\vec{\nabla}V\right|.\left|d\vec{r}\right|\cos\theta \tag{2.5 (b)}$$

where θ is the angle between $\vec{\nabla}V$ and $d\vec{r}$. So, if we fix the magnitude of $d\vec{r}$ and vary θ, then the maximum change of V will occur when $\theta = 0$. Hence for a fixed distance $d\vec{r}$, dV is greatest when one moves in the same direction as $\vec{\nabla}V$. So, $\vec{\nabla}V$ points along the direction of maximum increase of V. This is the effective meaning of gradient and the surfaces situated at fixed distances from the origin are known as the equi-potential surfaces.

For, any conservative force the general characteristics are as follows:

a) work done by a conservative force does not depend on the path followed in the force field,
b) the total work done over a closed path is zero in the conservative field and
c) the magnitude of the conservative force may be taken as a gradient of a scalar function.

Work energy theorem: If the total external force that acts on a particle is a function of its position *i.e.*, $\vec{F} = F(\vec{r})$, then the work done due to the force on the particle in moving it from its initial position to some final position is given by the line integral,

$$W_{if} = \int_i^f \vec{F}.d\vec{r} \tag{2.6}$$

This integral is known as the path integral of the force and represents the cumulative effect of a force over the whole space. We can show, by evaluating the integral that it produces a change in kinetic energy (KE) of the particle on which it acts.

Let us now evaluate the path integral of force.

$$W_{if} = \int_i^f \vec{F}.d\vec{r} = \int_i^f m\frac{d\vec{v}}{dt}.d\vec{r} = \int_i^f m\vec{v}.d\vec{v} = \int_i^f mv.dv = \frac{m}{2}\int_i^f d\left(v^2\right)$$

or,
$$W_{if} = \frac{m}{2}\left(v_f^2 - v_i^2\right) = T_f - T_i \tag{2.7}$$

This implies that the work done is equal to the difference in the kinetic energies of the particle between the initial and final positions. Eq. (2.7) is known as the work-energy theorem.

(i) If $T_f > T_i$; $W_{if} > 0$ and the work done is done by the force on the particle and its KE increases.

(ii) If however $T_f < T_i$; $W_{if} < 0$, *i.e.*, the work done is done by the particle against the force and its KE decreases.

(iii) When $T_f = T_i$; the speed of the particle is constant, then $W_{if} = 0$ and no change in its KE will occur.

Due to this reason, in any uniform circular motion the centripetal force does not work since there the force and the velocity of the particle at every point of motion are at perpendicular to each other. A force acting perpendicular to the motion only makes a change in the direction and not the magnitude of the velocity and hence always there is zero work done as in case (iii) above.

2.2.2 Non-conservative Forces

If the total mechanical energy of a force field is not a constant then the force field is called a non-conservative force field and the force acting on that force is known as the *non-conservative force*. For the non-conservative force the work done in displacing a particle from one point to the other depends on the path along which the particle is displaced but is independent of the initial and final positions, *e.g.*, frictional force, viscous force *etc*.

A body may be acted upon by both the conservative and non-conservative forces. One such example is the sliding of a body down an inclined plane with friction under gravity, where the friction is non conservative force and gravity is a conservative force. So, the total force acting on the body is given by,

$$\vec{F} = \vec{F}_F + \vec{F}_G$$

where $\vec{F}_G$ and $\vec{F}_F$ respectively denotes the conservative force due to gravity and dissipative forces due to friction.

The work energy theorem is true for all kind of forces: conservative or dissipative. So, the total work done by the total force when the body moves from some initial position to a final position is given by,

$$W_{if} = \int_i^f \vec{F}.d\vec{r} = \int_i^f \vec{F}_F.d\vec{r} + \int_i^f \vec{F}_G.d\vec{r} = (V_i - V_f) + W_{if}^F$$

where V is the potential function associated with the conservative force $\vec{F}_G$ and W_{if}^F is the work done due to dissipative force $\vec{F}_F$. So, the well known 'work energy theorem' assumes the following form,

$$\frac{1}{2}m\left(v_f^2 - v_i^2\right) = (V_i - V_f) + W_{if}^F$$

or, $$\left(\frac{1}{2}mv_f^2 + V_f\right) - \left(\frac{1}{2}mv_i^2 + V_i\right) = W_{if}^F$$

or, $$\left(T_f + V_f\right) - \left(T_i + V_i\right) = W_{if}^F$$

or, $$E_f - E_i = W_{if}^F \tag{2.8}$$

where $E = T + V$ is total mechanical energy of the system which, in a dissipative system, is no longer constant but depends on the state of the system. The mechanical energy is conserved, only when $W_{if}^F = 0$, *i.e.*, when the dissipative forces will do no work.

2.3 POTENTIAL ENERGY

The potential energy (PE) of a particle may be considered as the energy stored in it owing to its position. In a conservative force field the PE of a particle is measured by the amount of work that the particle can do when it moves from its early position to some final position. Generally, the final position is taken to be situated at infinity. If the early position is represented by a vector $\vec{r}$ and its final position by a vector $\vec{r_0}$ we have the PE of the particle as,

$$U = \int_{\vec{r}}^{\vec{r_0}} \vec{F}.d\vec{r}$$

If the final position be at infinity then we have $\vec{r_0} = \infty$ and so,

$$U = \int_{\vec{r}}^{\infty} \vec{F}.d\vec{r}$$

Conversely if the initial position be at infinity and PE at infinity will be zero, then PE of the particle at a distance $\vec{r}$ will be,

$$U = -\int_{\infty}^{\vec{r}} \vec{F}.d\vec{r} \qquad (2.9)$$

Hence in the central force field (*e.g.*, gravitational field, electrostatic field *etc.*), energy of a particle at a specified point is given by the amount of work done to move it from infinity up to the point.

Usually, final position is taken as a point where no force acts on the particle. For example in the case of a spring, when its length is equal to its unstretched length, no force acts on the spring. At this condition, PE of the spring is zero. Again, in case of gravitational or electrostatic field, no force acts on the particle at infinity. So, in such a field, potential at infinity is taken to be zero. Sometimes, in the case of earth, PE on its surface is also taken to be as zero.

2.4 FORCE AND POTENTIAL RELATIONSHIP

In a conservative force field it can be shown that the force acting on a particle is equal to the negative gradient of the PE *i.e.*, $\vec{F} = -\vec{\nabla}V$. To show this let us take the PE of the particle at a distance $\vec{r}$ in a force field as given in Eq. (2.3) as,

$$U = -\int_{\infty}^{\vec{r}} \vec{F}.d\vec{r}$$

$$= -\int_{\infty}^{\vec{r}} (F_x dx + F_y dy + F_z dz)$$

Differentiating partially with respect to *x*, *y* and *z* we get,

$$F_x = -\frac{\partial U}{\partial x}$$

$$F_y = -\frac{\partial U}{\partial y}$$

and $$F_z = -\frac{\partial U}{\partial z}$$

So, $$\vec{F} = \hat{i}F_x + \hat{j}F_y + \hat{k}F_z = -\left(\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}\right) = -\vec{\nabla}U$$

Thus conservative force can be expressed as the negative gradient of the potential.

Now, $$\vec{\nabla}\times\vec{F} = -\vec{\nabla}\times\vec{\nabla}U = -\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{vmatrix}$$

or, $$\vec{\nabla}\times\vec{F} = \hat{i}\left(\frac{\partial^2 U}{\partial y\partial z} - \frac{\partial^2 U}{\partial z\partial y}\right) + \hat{j}\left(\frac{\partial^2 U}{\partial z\partial x} - \frac{\partial^2 U}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial^2 U}{\partial x\partial y} - \frac{\partial^2 U}{\partial y\partial x}\right)$$

or, $\vec{\nabla}\times\vec{F} = 0$ [as for U to be a perfect differential $\frac{\partial^2 U}{\partial u\partial v} = \frac{\partial^2 U}{\partial v\partial u}$]

So, the curl of conservative force vanishes.

EXAMPLE 2.1

Example 2.1 Show that the force $\vec{F} = \hat{i}\left(y^2z^3 - 6xz^2\right) + \hat{j}2xyz^3 + \hat{k}\left(3xy^2z^2 - 6x^2z\right)$ is conservative. If the force acts on a particle and displaces it from (0, 2, 1) to (2, 5, 4), calculate the work done.

Solution

Here $$\vec{F} = \hat{i}\left(y^2z^3 - 6xz^2\right) + \hat{j}2xyz^3 + \hat{k}\left(3xy^2z^2 - 6x^2z\right)$$

So, $$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 - 6xz^2 & 2xyz^3 & 3xy^2z^2 - 6x^2z \end{vmatrix}$$

or, $$\vec{\nabla}\times\vec{F} = \hat{i}\left[\frac{\partial}{\partial y}\left(3xy^2z^2 - 6x^2z\right) - \frac{\partial}{\partial z}\left(2xyz^3\right)\right] + \hat{j}\left[\frac{\partial}{\partial z}\left(y^2z^3 - 6xz^2\right) - \frac{\partial}{\partial x}\left(3xy^2z^2 - 6x^2z\right)\right] + \hat{k}\left[\frac{\partial}{\partial x}\left(2xyz^3\right) - \frac{\partial}{\partial y}\left(y^2z^3 - 6xz^2\right)\right]$$

or, $\vec{\nabla}\times\vec{F} = 0$; so the force field is conservative.

When the force $\vec{F} = \hat{i}\left(y^2z^3 - 6xz^2\right) + \hat{j}2xyz^3 + \hat{k}\left(3xy^2z^2 - 6x^2z\right)$ acts on a particle and displaces it from (0, 2, 1) to (2, 5, 4), the work done will be,

$$W = \int_{(0,2,1)}^{(2,5,4)} \vec{F}.d\vec{r} = \int_{(0,2,1)}^{(2,5,4)} (F_x dx + F_y dy + F_z dz)$$

or, $$W=\int_{(0,2,1)}^{(2,5,4)} d\left(xy^2z^3-3x^2z^2\right)$$

or, $$W=\left(xy^2z^3-3x^2z^2\right)\Big|_{(0,2,1)}^{(2,5,4)}$$

$$=2\times5^2\times4^3-3\times2^2\times4^2-0$$

$$=3200-192=3008 \text{ unit.}$$

EXAMPLE 2.1

Example 2.2 Show that the force $\vec{F}=\hat{i}\left(2xy+z^2\right)+\hat{j}x^2+\hat{k}2xz$ is conservative. Hence find the corresponding potential.

Solution

Here $$\vec{F}=\hat{i}\left(2xy+z^2\right)+\hat{j}x^2+\hat{k}2xz \qquad \text{(a)}$$

So, $$\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2xy+z^2 & 2xyz^3 & 2xz\end{vmatrix}$$

or, $$\vec{\nabla}\times\vec{F}=\hat{i}\left[\frac{\partial}{\partial y}(2xz)-\frac{\partial}{\partial z}\left(x^2\right)\right]+\hat{j}\left[\frac{\partial}{\partial z}(2xy+z)-\frac{\partial}{\partial x}(2xz)\right]$$
$$+\hat{k}\left[\frac{\partial}{\partial x}\left(x^2\right)-\frac{\partial}{\partial y}\left(2xy+z^2\right)\right]$$

or, $$\vec{\nabla}\times\vec{F}=0$$

So the force field is conservative.

Now for a conservative force field we have,

$$\vec{F}=-\vec{\nabla}V=-\hat{i}\frac{\partial V}{\partial x}-\hat{j}\frac{\partial V}{\partial y}-\hat{k}\frac{\partial V}{\partial z} \qquad \text{(b)}$$

Comparing (a) and (b) we get,

$$\frac{\partial V}{\partial x}=-\left(2xy+z^2\right) \quad \text{or, } V=-x^2y-xz^2+C_1$$

$$\frac{\partial V}{\partial y}=-x^2 \quad \text{or, } V=-x^2y+C_2$$

$$\frac{\partial V}{\partial z}=-2xz \quad \text{or, } V=-xz^2+C_3$$

So, combing all the possible values of potential we get,

$$V=-x^2y-xz^2+C$$

which gives the required potential.

EXAMPLE 2.2

2.5 CENTRAL FORCE

A *central force* is a force which at every point of its field is directed either towards a fixed point or away from a fixed point (the centre of force) in the field and the magnitude of which at a point is a function of the distance (r) of the point from the fixed origin only. The path traced out by a particle under the action of a central force is called a *central orbit*.

The central force may be mathematically represented as,

$$\vec{F}(\vec{r}) = F(r)\hat{r} \tag{2.10}$$

where $\hat{r}$ is the unit vector along the position vector $\vec{r}$ and $F(r)$ is a function of r only.

2.5.1 Characteristics of Motion

A particle in motion under central force has a number of important characteristics as stated below.

i) A central force is conservative in nature.

ii) The angular momentum is a constant of motion.

iii) The motion is planar, *i.e.*, the motion of the particle is confined to a fixed plane.

iv) The areal velocity of the radius vector *i.e.*, the line joining the particle with the centre is constant.

2.5.2 Nature of Orbit

As, the angular momentum is the moment of the corresponding linear momentum we have,

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \tag{2.11}$$

$\therefore$ $\vec{r} \times m\vec{v}$ = a constant vector in central motion.

The direction of $\vec{L}$ is thus always normal to the plane containing the vector $\vec{r}$ and $\vec{v}$.

Hence the path of the particle in a central force field must always lie in one fixed plane, the plane containing the vector $\vec{r}$ and $\vec{v}$.

EXAMPLE 2.3

Example 2.3 The position vector for a particle of mass 10 g is $5\hat{i} + 2\hat{j}$ cm and its velocity is $3\hat{i}$ cm/s. Calculate the angular momentum about the origin.

Solution

Angular momentum will be,

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = m\vec{r} \times \vec{v} = 10\left(5\hat{i} + 2\hat{j}\right) \times 3\hat{i} = 60\hat{k} \text{ g-cm}^2\text{/s.}$$

2.5.3 Conservation of Angular Momentum

Let O be the fixed point to which the force is always directed, $\vec{r}$ the position vector of the particle P at any instant t (Fig. 2.2). Let, $\vec{v}$ be the particle velocity which is acting tangentially to the orbit. So, the applied torque on the particle at that instant is,

$$\vec{N} = \vec{r} \times \vec{F} = \vec{r} \times \hat{r} F(r) = 0 \qquad (2.12)$$

where $\vec{F}$ is acting along the direction of $\vec{r}$. But, since the torque $\vec{N} = \frac{d\vec{L}}{dt}$, where $\vec{L}$ represents angular momentum vector of the particle, we have

$$\frac{d\vec{L}}{dt} = 0$$

So, $\vec{L} =$ constant

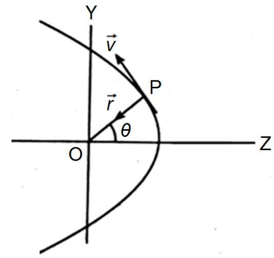

Fig. 2.2: Angular momentum conservation

Thus angular momentum vector ($\vec{L}$) of a particle in a central force field always remains constant and hence it is a constant of motion.

2.5.4 Areal Velocity

Let m be the mass of the particle and $\vec{r}$ the instantaneous position vector at P with the centre of force O as the origin. Let the position vector changes to $\vec{r} + d\vec{r}$ at Q in time dt.

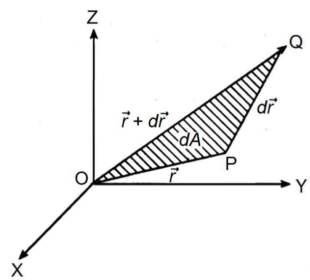

Fig. 2.3: Areal velocity

$\therefore$ $\qquad \text{PQ} = \left(\vec{r} + d\vec{r}\right) - \vec{r} = d\vec{r}$

The area swept by the radius vector $\vec{r}$ in time dt is the area of ΔOPQ [Fig. 2.3] and is given by,

$$d\vec{A} = \frac{1}{2}\vec{r} \times d\vec{r}$$

The rate at which the radius vector sweeps the area is defined as the *areal velocity*. So, here the areal velocity of the mass motion will be,

$$\frac{d\vec{A}}{dt} = \frac{1}{2}\vec{r} \times \frac{d\vec{r}}{dt} = \frac{1}{2m}\vec{r} \times m\frac{d\vec{r}}{dt} = \frac{1}{2m}\vec{r} \times m\vec{v} = \frac{\vec{L}}{2m} = \text{constant} \qquad (2.13)$$

Since $\vec{L}$ is a constant of motion and m is necessarily constant, so under a central force the areal velocity of the radius vector of a particle is a constant.

2.6 GENERAL EQUATION OF A CENTRAL ORBIT

We have already explained that the motion of a particle under central force is confined to a plane. It may thus be conveniently represented by plane polar coordinates (r, θ). Since the force is radial, being always directed to (or away from) the centre, the radial acceleration f_r only exists and cross-radial acceleration f_θ is zero.

$\therefore$ We have $\qquad f_\theta = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \qquad (2.14)\text{ (a)}$

and $\qquad f_r = \ddot{r} - r\dot{\theta}^2 = \pm\frac{F(r)}{m} \qquad (2.14)\text{ (b)}$

where $F(r)$ is the magnitude of the radial force. The negative sign corresponds to force directed to the origin and positive sign corresponds to force directed away from the origin. Now from Eq. (2.14) (a) we have, $r^2 \dot{\theta} = \text{constant} = h$ (say)

Introducing for convenience, the reciprocal coordinate u defined as $r = \frac{1}{u}$ we have,

$$\dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -h\frac{du}{d\theta}$$

and
$$\ddot{r} = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -h^2u^2\frac{d^2u}{d\theta^2}$$

$\therefore$ We have from Eq. (2.8)

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}.\frac{h^2}{r^4} = \pm\frac{F(r)}{m}$$

or,
$$\frac{d^2u}{d\theta^2} + u = \mp\frac{F(1/u)}{mh^2u^2}$$

or,
$$\frac{d^2u}{d\theta^2} + u = \mp\frac{f(1/u)}{h^2u^2} \qquad (f = F/m = \text{the acceleration}) \qquad (2.15)\ (a)$$

Since $h = r^2\dot{\theta}, mh = mr^2\dot{\theta} =$ the angular momentum L, Thus, using this the above equation can be written as,

$$\frac{d^2u}{d\theta^2} + u = \mp\frac{mF(1/u)}{L^2u^2} \qquad (2.15)\ (b)$$

This is the alternative form of the differential equation of the orbit.

2.7 ENERGY IN CENTRAL FORCE FIELD

If any force acting on a particle has the following two properties, viz., (i) from a fixed point the force is always directed toward or away and (ii) the magnitude of the applied force depends only on the distance from the centre of force, then the forces having these two properties are known as central forces.

Let, a particle of mass m is acted upon by a central force and moves with an angular momentum L. (r,θ) being its polar co-ordinates. Introducing the reciprocal co-ordinate $r = \frac{1}{u}$, we get

$$\dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -\frac{L}{m}\frac{du}{d\theta} \qquad \left(\because r^2\dot{\theta} = \frac{L}{m}\right) \qquad (2.16)$$

Now, the total mechanical energy of the particle in plane polar coordinates is

$$E = \frac{1}{2}mv^2 + V(r) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r), \text{ where } V(r) \text{ is the PE.}$$

$$\therefore \qquad E=\frac{1}{2}m\left[\left(-\frac{L}{m}\frac{du}{d\theta}\right)^2+r^2\left(\frac{L}{mr^2}\right)^2\right]+V(r)=\frac{L^2}{2m}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]+V(r) \qquad (2.17)$$

which is the required expression for the energy in a central force field.

2.7.1 Conservation of Energy in Central Motion

In plane polar coordinates, the radial and the cross-radial accelerations are given respectively by

$$f_r=\ddot{r}-r\dot{\theta}^2 \quad \text{and } f_\theta=r\ddot{\theta}+2\dot{r}\dot{\theta}$$

$$\therefore \qquad F(r)=m\left(\ddot{r}-r\dot{\theta}^2\right)=\text{ radial force}$$

and $$F(\theta)=m\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)=\text{cross radial force}$$

Now, the central motion is radial so that only $F(r)$ exists and $F(\theta)=0$.

To prove this let us consider the magnitude of the angular momentum $L=\left|\vec{L}\right|=mr^2\dot{\theta}$.

$$\therefore \qquad \frac{d\vec{L}}{dt}=2mr\dot{r}\dot{\theta}+mr^2\ddot{\theta}=mr\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)=0$$

Since $\vec{L}$ is a constant of motion in a central force field, substituting $\dot{\theta}=\dfrac{L}{mr^2}$ we obtain

$$F(r)=m\ddot{r}-\frac{L^2}{mr^3}$$

Also, a central force is conservative and may thus be expressed as the negative gradient of potential (energy) function V as, $F(r)=-\vec{\nabla}V$.

$$\therefore \qquad m\ddot{r}-\frac{L^2}{mr^3}=F(r)=-\frac{dV}{dr}$$

or, $$m\ddot{r}=-\frac{d}{dr}\left(V+\frac{L^2}{2mr^2}\right)$$

Multiplying both sides of the above relation by $\dot{r}$, we get

$$m\dot{r}\ddot{r}=-\frac{d}{dr}\left(V+\frac{L^2}{2mr^2}\right)\dot{r}$$

or, $$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right)=-\frac{d}{dr}\left(V+\frac{L^2}{2mr^2}\right)\frac{dr}{dt}=-\frac{d}{dt}\left(V+\frac{L^2}{2mr^2}\right)$$

or, $$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2+V+\frac{L^2}{2mr^2}\right)=0$$

or, $$\frac{1}{2}m\dot{r}^2 + V + \frac{L^2}{2mr^2} = E = \text{const.} \tag{2.18}$$

or, $$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V = \text{const.} = E \qquad (\because L = mr^2\dot{\theta})$$

or, $$\frac{1}{2}mv^2 + V = \text{const.}$$

So, $$T + V = \text{const.}$$

where $T = \frac{1}{2}mv^2 = \text{KE}$.

Thus, in a central field the total energy $E = T + V$ is another constant of motion and conservation of energy holds in central motion.

EXAMPLE 2.4

Example 2.4 A planet revolves around a star in an elliptic Orbit. The ratio of longest distance to the shortest of the planet from the star is 4. Find the ratio of KE of the planet at longest to the closest position.

Solution

Gravitational force is central and hence the angular momentum L of the planet remains constant during its revolution.

Kinetic energy $$T = \frac{L^2}{2mr^2}$$

or, $$T \propto \frac{1}{r^2}$$

So, $$\frac{T_2}{T_1} = \frac{r_1^2}{r_2^2}$$

or, $$\frac{T_2}{T_1} = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

This is the ratio of the kinetic energies of the planet at the farthest to the closest position.

2.8 REDUCTION OF TWO-BODY PROBLEM

Here we will show the way of motion of two different bodies interacting in a gravitational force field which is equivalent to the movement of one body with an equivalent reduced mass. Consider two bodies having masses m_1 and m_2 interact with each other such that by Newton's third law the force on body 2 due to body 1 is equal in magnitude and opposite in direction to the force on body 1 due to body 2 (Fig. 2.4). Mathematically it is given by,

$$\vec{F}_{12} = -\vec{F}_{21} \tag{2.19}$$

Now, applying Newton's second law to the individual bodies we get,

$$\vec{F}_{12} = m_1 \frac{d^2 \vec{r}_1}{dt^2}$$

and $$\vec{F}_{21} = m_2 \frac{d^2 \vec{r}_2}{dt^2}$$

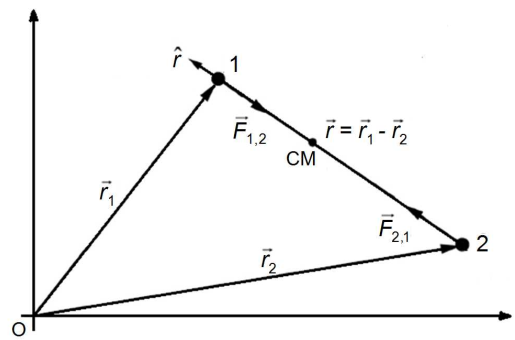

Fig. 2.4: Two body problem

Rearranging we get,

$$\frac{\vec{F}_{12}}{m_1} = \frac{d^2 \vec{r}_1}{dt^2} \tag{2.20}$$

and $$\frac{\vec{F}_{21}}{m_2} = \frac{d^2 \vec{r}_2}{dt^2} \tag{2.21}$$

Subtracting Eq. (2.21) from Eq. (2.20) we get,

$$\frac{\vec{F}_{12}}{m_1} - \frac{\vec{F}_{21}}{m_2} = \frac{d^2 \vec{r}_1}{dt^2} - \frac{d^2 \vec{r}_2}{dt^2}$$

$$= \frac{d^2 \vec{r}}{dt^2} \tag{2.22}$$

Using Eq. (2.19), Eq (2.22) becomes

$$\vec{F}_{12}\left(\frac{1}{m_2} + \frac{1}{m_2}\right) = \frac{d^2 \vec{r}}{dt^2}$$

or, $$\frac{\vec{F}_{12}}{\mu} = \frac{d^2 \vec{r}}{dt^2}$$

where μ is the reduced mass and is given by

$$\frac{1}{\mu} = \frac{1}{m_2} + \frac{1}{m_2}$$

or, $$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{2.23}$$

So $$\vec{F}_{12} = \mu \frac{d^2 \vec{r}}{dt^2} \tag{2.24}$$

This expression for $\vec{F}_{12}$ can be interpreted specially using Newton's second law. In Eq. (2.24) μ be the reduced mass of a reduced object with position vector $\vec{r}$ with respect to the origin O due to an attractive gravitational force directed toward the origin.

So, the basic two-body gravitational problem is reduced to single-body problem with reduced mass μ by the influence of a central force $-\vec{F}_{12}$ with $\vec{r}$ as the distance between the reduced body and the origin.

Here in this reformulation, not a body is located at the origin. Whereas, $\vec{r}$ in the two-body problem is the relative distance between the original two bodies.

2.9 ENERGY AND ANGULAR MOMENTUM

Consider the reduced body with reduced mass, orbiting about a fixed point under the influence of a radially attractive force. About the origin O both the energy (E) and the angular momentum (L) of the motion for the equivalent single-body problem are constants. E is a constant because there are no external forces acting on the reduced body, and L about O is constant because the only force is directed towards the origin, and hence the torque about the origin due to that force is zero (Fig. 2.5). Since, L is a constant; the orbit of the reduced body is situated in a plane with L vector pointing at right angle to this plane.

Consider (r, θ) as the polar coordinates of the reduced body with respect to O and note that,

$$\vec{F}_{gravity} = -\vec{F}_{12} \tag{2.25}$$

Since the force is conservative, the PE is given by $U(r) = -\dfrac{Gm_1m_2}{r}$ with the choice of potential at infinity as $U(\infty) = 0$.

The total energy is the sum of the KE and the PE and is given by

$$E = \frac{1}{2}\mu v^2 - \frac{Gm_1m_2}{r} \tag{2.26}$$

where KE is, $\frac{1}{2}\mu v^2$, μ is reduced mass and v is the relative speed of the two bodies and is given by

$$v^2 = v_r^2 + v_\theta^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2$$

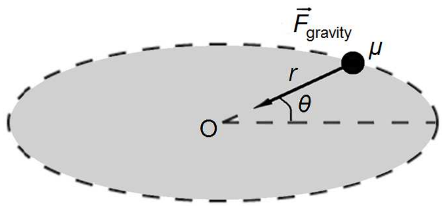

Fig. 2.5: Motion of the reduced body in polar coordinate system

here $v_r = \dfrac{dr}{dt}$ is the radial component of velocity and $v_\theta = r\dfrac{d\theta}{dt}$ is the cross-radial component of velocity.

So,
$$E = \frac{1}{2}\mu\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] - \frac{Gm_1m_2}{r} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{Gm_1m_2}{r} \tag{2.27}$$

The magnitude of the angular momentum with respect to the centre of mass will be

$$L = \mu r v_\theta = \mu r^2\frac{d\theta}{dt} = \mu r^2\dot{\theta} \tag{2.28}$$

EXAMPLE 2.5

Example 2.5 A particle follows a spiral orbit is given by $r = c\theta^2$ under an unknown force. Prove that such an orbit is possible in a central field. Also find the form of the force law.

Solution

The equation of spiral orbit is given by,

$$u = \frac{1}{r} = \frac{1}{c\theta^2}$$

where u is the reciprocal co-ordinate.

So, $$\frac{du}{d\theta} = -\frac{2}{c\theta^3} \text{ and } \frac{d^2u}{d\theta^2} = \frac{6}{c\theta^4}$$

Under central force, the differential equation of the orbit is,

$$\frac{d^2u}{d\theta^2} + u = -\frac{mF(1/u)}{L^2u^2}$$

or, $$F(1/u) = -\frac{L^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right)$$

or, $$F(1/u) = -\frac{L^2u^2}{m}\left(\frac{6}{c\theta^4} + \frac{1}{c\theta^2}\right)$$

or, $$F(r) = -\frac{L^2}{mr^2}\left(\frac{6}{c\theta^4} + \frac{1}{c\theta^2}\right)$$

or, $$F(r) = -\frac{L^2}{m}\left[\frac{6c}{\left(c\theta^2\right)^4} + \frac{1}{\left(c\theta^2\right)^3}\right] = -\frac{L^2}{m}\left[\frac{6c}{r^4} + \frac{1}{r^3}\right]$$

which is the required form of the force law. As the force $F(r)$ depends on r only, it represents a central force.

EXAMPLE 2.5

2.10 EQUATION FOR THE ORBIT OF REDUCED BODY

From Eq. (2.15) we get the equation for the orbit of the reduced body as

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu G m_1 m_2}{L^2} \tag{2.29}$$

Eq. (2.29) is mathematically equivalent to the harmonic oscillator equation with a constant at the right hand side. The solution of Eq. (2.29) will have two parts:

i) the angle-independent inhomogeneous solution $u_0 = \frac{\mu G m_1 m_2}{L^2}$ representing a circular orbit,

ii) a sinusoidally varying homogeneous solution like $u_1(\theta) = A\cos(\theta - \theta_0)$; where A and θ_0 are constants which can be evaluated from the nature of the orbit.

As $$u_0 = \frac{1}{r_0}$$

$\therefore$ $$r_0 = \frac{L^2}{\mu G m_1 m_2} \tag{2.30}$$

This is called the semi latus rectum of the orbit. The complete solution of Eq. (2.29) may thus be

$$u = u_0 + u_1 = \frac{1}{r_0} + A\cos(\theta - \theta_0) = \frac{1}{r_0}\left[1 + r_0 A\cos(\theta - \theta_0)\right] \tag{2.31 (a)}$$

or, $$r = \frac{r_0}{1 + r_0 A\cos(\theta - \theta_0)} \tag{2.31 (b)}$$

Let us choose $A = \frac{\varepsilon}{r_0}$ (2.32)

where $$\varepsilon = \sqrt{1 + \frac{2\mu E r_0^2}{L^2}} = \sqrt{1 + \frac{2EL^2}{\mu(Gm_1m_2)^2}} \tag{2.33}$$

is the eccentricity and $\theta_0 = \pi$.

So, Eq. (2.31) (b) can be written as,

$$r = \frac{r_0}{1 - \varepsilon\cos\theta} \tag{2.34}$$

Again in terms of r_0 from Eq. (2.30) we have the angular momentum as,

$$L = (\mu G m_1 m_2 r_0)^{\frac{1}{2}} \tag{2.35}$$

and in terms of r_0 and ε, from Eq. (2.33) and Eq. (2.35) we have the energy as,

$$E = \frac{Gm_1m_2(\varepsilon^2 - 1)}{2r_0} \tag{2.36}$$

Eq. (2.34) represents a general conic section and with the aid of Cartesian coordinate transformation $x = r\cos\theta$ and $y = r\sin\theta$, along with $r^2 = x^2 + y^2$, the orbit equation can be rewritten as, $r = r_0 + \varepsilon r\cos\theta$

or, $(x^2 + y^2)^{\frac{1}{2}} = r_0 + \varepsilon x$

or, $x^2 + y^2 = r_0^2 + 2\varepsilon x r_0 + \varepsilon^2 x^2$

or, $$x^2(1 - \varepsilon^2) - 2\varepsilon x r_0 + y^2 = r_0^2 \tag{2.37}$$

This is the general expression of a conic section with axis along the X-direction. For a given $r_0 > 0$, corresponding to a given non-zero L as in Eq. (2.30), there are different cases determined by the value of ε.

Case 1: When $\varepsilon = 1, E = E_{min} < 0$ and $r = r_0$, Eq. (2.37) will be reduced to,

$x^2 + y^2 = r_0^2$; this represents the equation for a circle.

Case 2: When $0 < \varepsilon < 1, E_{min} < E < 0$, Eq. (2.37) describes an ellipse as given by,

$y^2 + Ax^2 - Bx = k$; where $A > 0$ and k is a positive constant.

Case 3: When $\varepsilon = 1, E = 0$, Eq. (2.37) describes a parabola as given by,

$$x = \frac{y^2}{2r_0} - \frac{r_0}{2}$$

Case 4: When $\varepsilon > 1, E > 0$, Eq. (2.37) describes a hyperbola as given by,

$y^2 - Ax^2 - Bx = k$; where $A > 0$ and k is a positive constant.

2.11 ENERGY EQUATION AND ENERGY DIAGRAMS

From Eq. (2.27), we have

$$E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{Gm_1m_2}{r} = \frac{1}{2}\mu v^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 - \frac{Gm_1m_2}{r} \qquad (2.38)\ (a)$$

Again, using Eq. (2.22), we have

$$L = \mu r^2\dot{\theta}.$$

So, the total energy will be,

$$E = \frac{1}{2}\mu v^2 + \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r} = K_{eff} + U_{eff} \qquad (2.38)\ (b)$$

where the effective PE is given by,

$$U_{eff} = \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r} \qquad (2.39)$$

and the effective KE is

$$K_{eff} = \frac{1}{2}\mu v^2 \qquad (2.40)$$

The variation of U_{eff} with r, is shown in Fig. 2.6. The extreme upper curve varies as $\frac{1}{r^2}$ and the extreme lower varies as $-\frac{1}{r}$. The sum U_{eff} is represented by the middle curve.

The minimum value of U_{eff} will occur at $r = r_0$. Whenever $K_{eff} = 0$, the total energy is equal to the effective PE,

$$E = U_{eff} = \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r} \qquad (2.41)$$

Now, the PE is defined to be the negative integral of the work done by the force.

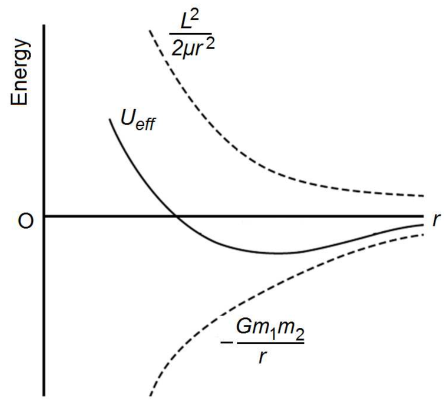

Fig. 2.6: Variation of effective potential energy

For the reduction to the one-body problem, using this U_{eff}, we can introduce an effective force $\vec{F}_{eff}$ such that $U^B_{eff} - U^A_{eff} = -\int_A^B \vec{F}_{eff}.d\vec{r} = -\int_A^B (F_r)_{eff}.dr$

Again, $U^B_{eff} - U^A_{eff} = \int_A^B \frac{dU_{eff}}{dr}dr$

Comparing we find that the radial component of $\vec{F}_{eff}$ is the negative of the derivative U_{eff}, *i.e.*,

$$(F_r)_{eff} = -\frac{dU_{eff}}{dr} \tag{2.42}$$

The resulting PE gives the PE for a reduced body travelling in single dimension. It is a function of r only and does not depend on θ.

$$(F_r)_{eff} = -\frac{dU_{eff}}{dr} = -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r}\right) = \frac{L^2}{\mu r^3} - \frac{Gm_1m_2}{r^2} \tag{2.43}$$

So, there are two forces acting on the reduced body, $(F_r)_{eff} = (F_r)_g + (F_r)_c$

where the effective centrifugal force is $(F_r)_c = \dfrac{L^2}{\mu r^3}$

and the gravitational force is $(F_r)_g = -\dfrac{Gm_1m_2}{r^2}$.

2.11.1 Circular, Elliptic, Parabolic and Hyperbolic Orbits

Case 1: Circular Orbit ($E = E_{\min}$)

The lowest energy state, $E_{\min}$, corresponds to the minimum of the effective PE, *i.e.*,

$$E_{\min} = (U_{eff})_{\min}$$

When this condition is satisfied the effective KE is zero *i.e.*,

$$K_{eff} = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 = 0$$

Hence, the radial velocity v_r is zero, so the distance r from the central point is a constant, which is the condition for a circular orbit. The condition for the minimum of the effective PE is,

$$0 = \frac{dU_{eff}}{dr} = -\frac{L^2}{\mu r^3} + \frac{Gm_1m_2}{r^2}$$

which when solved we get,

$$r = r_0 = \frac{L^2}{Gm_1m_2} \tag{2.44}$$

Case 2: Elliptic Orbit $(E_{\min} < E < 0)$

When $K_{eff} = 0$, the mechanical energy is $E = U_{eff}$. Now, $\dfrac{dr}{dt} = 0$ corresponds to a point of closest or furthest approach (Fig. 2.6). This condition represents the maximum value as well as the minimum value of r for an orbit elliptical in nature.

Now, rewriting $E = \dfrac{L^2}{2\mu r^2} - \dfrac{Gm_1m_2}{r}$ we get the quadratic type of equation for the distance r as,

$$r^2+\frac{Gm_1m_2}{E}r-\frac{L^2}{2\mu E}=0 \tag{2.45}$$

The two roots of the equations are,

$$r=-\frac{Gm_1m_2}{2E}\pm\left\{\left(\frac{Gm_1m_2}{2E}\right)^2+\frac{L^2}{2\mu E}\right\}^{\frac{1}{2}}$$

or,
$$r=-\frac{Gm_1m_2}{2E}\left[1\pm\left\{1+\frac{2EL^2}{\mu\left(Gm_1m_2\right)^2}\right\}^{\frac{1}{2}}\right]$$

or,
$$r=-\frac{Gm_1m_2}{2E}(1\pm\varepsilon) \qquad \text{[using Eq. (2.33)]} \tag{2.46}$$

So,
$$\frac{r_0}{1-\varepsilon^2}=\frac{\dfrac{L^2}{\mu Gm_1m_2}}{1-\left[1+\dfrac{2L^2E}{\mu\left(Gm_1m_2\right)^2}\right]}=\frac{\dfrac{L^2}{\mu}}{-\dfrac{2L^2E}{\mu\left(Gm_1m_2\right)^2}}=-\frac{Gm_1m_2}{2E} \tag{2.47}$$

Substituting Eq. (2.47) into Eq. (2.46) we get an expression for the points of closest and furthest approach as,

$$r=\frac{r_0}{1-\varepsilon^2}(1\pm\varepsilon) \tag{2.48}$$

The plus sign corresponds to the distance of closest approach,

$$r=r_{\min}=\frac{r_0}{1+\varepsilon} \tag{2.49}$$

and the minus sign corresponds to the distance of furthest approach,

$$r=r_{\max}=\frac{r_0}{1-\varepsilon} \tag{2.50}$$

Case 3: Parabolic Orbit ($E = 0$)

If $E = 0$, the effective PE approaches zero ($U_{eff}\to 0$) at the situation when the distance r approaches to infinity ($r\to\infty$). Since the total energy is zero, when $r\to\infty$ the KE also approaches zero, $K_{eff}\to 0$. This corresponds to a parabolic orbit. Now, for a body to escape from a planet, the body must have a total energy $E = 0$ (we set $U_{eff} = 0$ at infinity). This escape velocity condition corresponds to a parabolic orbit.

For a parabolic orbit, the body also has a distance of closest approach. This distance r_{par} can be obtained by solving

$$E=U_{eff}=\frac{L^2}{2\mu r^2}-\frac{Gm_1m_2}{r}=0 \tag{2.51}$$

as, $$r_{par} = \frac{L^2}{2\mu G m_1 m_2} = \frac{1}{2r_0} \tag{2.52}$$

the fact that the minimum distance to the origin *i.e.*, the focus of a parabola is half the semi-latus rectum is one fundamental property of a parabola.

Case 4: Hyperbolic Orbit ($E > 0$)

When $E > 0$, in the limit $r \to \infty$, $K_{eff} > 0$. This corresponds to a hyperbolic orbit. Here the closest approach condition is similar to Eq. (2.51) except E is now positive suggesting only one positive solution of Eq. (2.45). The distance of closest approach for this hyperbolic orbit will be,

$$r_{hyp} = \frac{r_0}{1+\varepsilon} \tag{2.53}$$

The constant r_0 is independent of the energy and from Eq. (2.33) as the reduced body energy increases, there will be an increase of eccentricity and so the distance of nearest approach r_{hyp} reduces.

2.12 LAWS OF PLANETARY MOTION

Kepler studied the motion of the planets in great detail and formulated his findings in the form of three important laws, known as the laws of planetary motion. These laws give a rather simple but accurate description of the planetary motion and it is by way of explaining the laws that Newton discovered his famous law of gravitation.

2.12.1 Kepler's First Law

All planets move in elliptic orbits round the sun that occupies one of the foci of the ellipse.

When $E < 0$, from Eq. (2.33) it is obvious that, $0 \le \varepsilon < 1$. These orbits are either circles or ellipses. Elliptic orbit laws are true only when there is single central force and the gravitational interactions owing to other bodies in the universe have no importance.

2.12.2 Kepler's Second Law

From the sun to any planet, the radius vector sweeps out same areas in the same interval of time. Now, the sum of the areas of the triangles in Fig. 2.7 in the limit of small $\Delta\theta$ is given by,

$$\Delta A = \frac{1}{2}(r\Delta\theta)r + \frac{r\Delta\theta}{2}\Delta r$$

So, the average rate of the change of this area in time Δt is,

$$\frac{\Delta A}{\Delta t} = \frac{1}{2}\frac{r\Delta\theta}{\Delta t}r + \frac{r\Delta\theta}{2}\frac{\Delta r}{\Delta t}$$

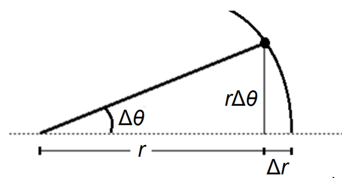

Fig. 2.7: Illustration of Kepler's 2nd law

As $\Delta t \to 0, \Delta\theta \to 0$, and

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \tag{2.54}$$

Again we have, $$\frac{d\theta}{dt} = \frac{L}{\mu r^2}$$

So, $$\frac{dA}{dt} = \frac{L}{2\mu} \tag{2.55}$$

As L and μ are constants, the time rate of change of area is a constant *i.e.*, equal areas are swept out in equal interval of time.

2.12.3 Kepler's Third Law

Square of the planet's revolution time period is proportional to the cube of the semi-major axis of the elliptic orbit $\left(T^2 \propto a^3\right)$.

From Eq. (2.55) we can write it in the form, $2\mu \frac{dA}{dt} = L$

Integrating we get, $\int_C 2\mu dA = \int_0^T L dt$

or, $$T = \frac{2\pi\mu ab}{L} \tag{2.56}$$

where T is the period of the orbit. For an elliptical orbit, $A = \pi ab$; where a and b are the semi-major axis and semi-minor axis respectively.

Now, $$T^2 = \frac{4\pi^2 \mu^2 a^2 b^2}{L^2}$$

Again using Eq. (2.35) we have,

$$L^2 = \mu G m_1 m_2 r_0 = \mu G m_1 m_2 a(1-\varepsilon^2)$$

Hence, $$T^2 = \frac{4\pi^2 \mu^2 a^2 b^2}{\mu G m_1 m_2 a\left(1-\varepsilon^2\right)}$$

SCAN ME

for more on Kepler's law

Again, the semi-minor axis is given by

$$b = \sqrt{\frac{aL^2}{\mu G m_1 m_2}}$$

Hence, $$T^2 = \frac{4\pi^2 \mu^2 a^3}{\mu G m_1 m_2} = \frac{4\pi^2 a^3}{G(m_1 + m_2)} \quad \text{[using Eq. (2.23)]} \tag{2.57}$$

2.13 SATELLITE MANEUVERS

We consider that a satellite of mass m_s is initially in a circular orbit of radius r around the earth of mass M_e and radius R_e. Let $M_e >> m_s$.

The magnitude of the acceleration for the circular orbit is $\frac{v^2}{r}$, and so the force equation for the motion of the satellite will be obtained by balancing the gravitational force with the centripetal force.

Thus, $$\frac{m_s v^2}{r} = \frac{Gm_s M_e}{r^2}$$

So, $$v = \sqrt{\frac{GM_e}{r}} \tag{2.58}$$

Hence, the total mechanical energy is,

$$E = -\frac{GM_e m_s}{r} + \frac{1}{2} m_s v^2 = -\frac{GM_e m_s}{2r} \qquad [\text{Let } U(r) \to 0 \text{ as } r \to \infty] \tag{2.59}$$

Let, us now consider that the satellite trajectory is changed to an elliptical orbit as a result of an orbital maneuver. This can be done by rocket firing for an interval of short time and thereby enhancing the satellite's tangential speed. Let, the farthest distance from earth also known as apogee of the elliptical orbit is three times the closest approach *i.e.*, perigee of the elliptical orbit. So,

$$r_a = 3r_p = 3r$$

Now, from the conservation of the angular momentum we have, $r_p v_p = r_a v_a$

or, $$rv_p = 3rv_a$$

So, $$v_a = v_p/3$$

The condition that the mechanical energy is constant then becomes,

$$\frac{1}{2} m_s v_p^2 - \frac{Gm_s M_e}{r} = \frac{1}{2} m_s v_a^2 - \frac{Gm_s M_e}{r_a} = \frac{1}{2} m_s \left(\frac{v_p}{3}\right)^2 - \frac{Gm_s M_e}{3r}$$

or, $$\frac{4}{9} v_p^2 = \frac{2GM_e}{3r}$$

or, $$v_p = \sqrt{\frac{3GM_e}{2r}} \tag{2.60}$$

So, $v_p > v$.

After burning of the rocket, the final mechanical energy is,

$$E_f = -\frac{GM_e m_s}{r} + \frac{1}{2} m_s v_p^2$$

$$= -\frac{GM_e m_s}{r} + \frac{3GM_e m_s}{4r}$$

or, $$E_f = -\frac{Gm_s M_e}{4r} = -\frac{Gm_s M_e}{A} \tag{2.61}$$

where $A = r + 3r$ is the major axis of the ellipse.

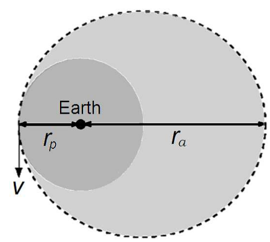

Fig. 2.8: Satellite maneuvers

UNIT SUMMARY

- ***Conservative and non-conservative forces***

$$\vec{\nabla}\times\vec{F}=0 \qquad \vec{F}=-\vec{\nabla}V$$

- ***Potential energy***

$$U=-\int_{\infty}^{\vec{r}}\vec{F}.d\vec{r}$$

- ***Relationship between conservative force and potential energy***

$$\vec{F}=-\vec{\nabla}V$$

- ***Central force: central orbit***

$$\vec{F}(\vec{r})=F(r)\hat{r}$$

- ***Characteristics of motion under central orbit***

i) A central force is conservative in nature.
ii) The angular momentum is a constant of motion.
iii) The motion is planar.
iv) The areal velocity of the radius vector is constant.

- ***General equation of a central orbit***

$$\frac{d^2u}{d\theta^2}+u=\mp\frac{mF(1/u)}{L^2u^2}$$

- ***Energy in a central force field***

$$E=\frac{1}{2}m\left[\left(-\frac{L}{m}\frac{du}{d\theta}\right)^2+r^2\left(\frac{L}{mr^2}\right)^2\right]+V(r)=\frac{L^2}{2m}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]+V(r)$$

- ***Conservation of energy in central motion***

$$\frac{1}{2}mv^2+V=\text{const.}$$

- ***Reduction of two-body problem into one-body problem***

$$\mu=\frac{m_1m_2}{m_1+m_2} \qquad \vec{F}_{12}=\mu\frac{d^2\vec{r}}{dt^2}$$

Energy and angular momentum

$$E=\frac{1}{2}\mu v^2-\frac{Gm_1m_2}{r}$$

$$L=\mu r v_\theta=\mu r^2\frac{d\theta}{dt}=\mu r^2\dot{\theta}$$

Equation for the orbit of the reduced body

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu Gm_1m_2}{L^2}$$

$$r = \frac{r_0}{1-\varepsilon\cos\theta}$$

Energy equation

$$E = \frac{1}{2}\mu v^2 + \frac{L^2}{2\mu r^2} - \frac{Gm_1m_2}{r} = K_{eff} + U_{eff}$$

Circular, elliptic, parabolic and hyperbolic orbits

Circular orbit ($E = E_{\min}$) $r = r_0 = \dfrac{L^2}{Gm_1m_2}$

Elliptic orbit $\left(E_{\min} < E < 0\right)$. $r = \dfrac{r_0}{1-\varepsilon^2}(1\pm\varepsilon)$

Parabolic orbit ($E = 0$) $r_{par} = \dfrac{L^2}{2\mu Gm_1m_2} = \dfrac{1}{2r_0}$

Hyperbolic orbit ($E > 0$) $r_{hyp} = \dfrac{r_0}{1+\varepsilon}$

- ***Kepler's laws of planetary motion***

  First law: All planets move in elliptic orbits round the Sun.

  Second law: Areal velocity $= \dfrac{dA}{dt} = \dfrac{L}{2\mu} =$ constant

  Third law: $T^2 \propto a^3$

- ***Satellite maneuvers***

$$E = -\frac{GM_em_s}{2r} \qquad v_p = \sqrt{\frac{3GM_e}{2r}} \qquad E_f = -\frac{Gm_sM_e}{4r}$$

EXERCISES

Multiple Choice Questions

2.1 What will be the time taken by a planet to sweep 2 million km^2 if the time taken by the same planet to cover 1 million km^2 is 36 hours?

(a) 18 hours (b) 36 hours (c) 72 hours (d) 144 hours

2.2 For circular planetary orbit the eccentricity is

(a) $\varepsilon > 1$ (b) $\varepsilon \geq 1$ (c) $\varepsilon < 1$ (d) $\varepsilon = 0$

2.3 When a planet orbits the Sun, one of the foci of the elliptical orbit should be at

(a) the axis (b) the perihelion (c) the centre (d) the Sun

2.4 Which of the following has an eccentricity of zero?

(a) a straight line (b) a large ellipse (c) a circle (d) a small ellipse

2.5 Kepler's second law is also called as

(a) the law of orbits (b) the law of areas (c) the law of periods (d) the law of gravity

2.6 Kepler's second law deals with
(a) the shape of the planet's orbits (b) the speed/area the planet travels
(c) the length of time it takes the planet to orbit the Sun
(d) time of travel

2.7 Kepler's third law is also called as
(a) the law of orbits (b) the law of areas (c) the law of periods (d) the law of gravity

2.8 Mathematical form of Kepler's third law is
(a) $T^3 = a^2$ (b) $T^3 = a^3$ (c) $T^2 = a^2$ (d) $T^2 = a^3$

2.9 Kepler's first law states that the orbits of the planets are oval in shape of
(a) ellipses (b) perfect circles (c) squares (d) triangles

2.10 The Kepler's first law is known as
(a) the law of orbits (b) the law of areas (c) the law of periods (d) the law of gravity

2.11 In a central force field, the angular momentum is
(a) zero (b) not conserved (c) infinity (d) conserved

2.12 The areal velocity of the particle in a central force field is
(a) zero (b) conserved (c) infinity (d) not conserved

2.13 At the turning point in an arbitrary potential field the radial velocity is
(a) zero (b) 1 (c) infinity (d) 1/2

2.14 For hyperbolic orbit the values of energy E and eccentricity ε are
(a) E = 0 and $\varepsilon > 1$ (b) E > 0 and $\varepsilon > 1$ (c) E > 0 and $\varepsilon = 1$ (d) E > 0 and $\varepsilon = 0$

2.15 For parabolic orbit the values of energy E and eccentricity ε are
(a) E = 0 and $\varepsilon = 1$ (b) E > 0 and $\varepsilon > 1$ (c) E > 0 and $\varepsilon = 1$ (d) E > 0 and $\varepsilon = 0$

2.16 For elliptical orbit the values of energy E and eccentricity ε are
(a) E = 0 and $\varepsilon > 1$ (b) E > 0 and $\varepsilon > 1$ (c) E < 0 and $\varepsilon < 1$ (d) E > 0 and $\varepsilon = 0$

2.17 A particle is moving on elliptical path under inverse square law force of the form $F(r) = -k/r^2$. The eccentricity of the orbit is
(a) a function of total energy (b) independent of total energy
(c) a function of angular momentum (d) independent of angular momentum

Answers of Multiple Choice Questions

2.1 (c), 2.2 (d), 2.3 (d), 2.4 (c), 2.5 (b), 2.6 (b), 2.7 (c), 2.8 (d), 2.9 (a), 2.10 (a), 2.11 (d), 2.12 (b), 2.13 (a), 2.14 (a), 2.15 (a), 2.16 (c), 2.17 (a) and (c).

Short and Long Answer Type Questions

Category I

2.1 Explain conservative force with examples.

2.2 Elaborate the idea of non-conservative force with examples.

2.3 With an example mention the role of central force.

2.4 Show that in case of conservative force work done over a closed path is zero.

2.5 Show that a conservative force can be represented by the relation $\vec{F} = -\vec{\nabla}V$ and curl of it is always zero.

2.6 Show that in non conservative force field mechanical energy is not conserved.

2.7 Show that for a non dissipative system $W_{if} = \int_i^f \vec{F}.d\vec{r} = T_f - T_i$ where T_f and T_i are the initial and the final kinetic energy.

2.8 Explain Kepler's second law of planetary motion.

2.9 Discuss the motion of a particle in a central force field and prove the conservation laws of linear momentum and total energy.

2.10 Prove that the total energy of a particle moving through a central force is a constant of motion.

2.11 Show that angular momentum of a particle moving in a central force field is conserved.

2.12 Derive the polar equation of elliptical orbit.

2.13 Derive the Kepler's third law of planetary motion.

Category II

2.14 Which force is required to obtain circular motion of the particle around the centre of the force?

2.15 Show that for a particle moving through inverse square law forces, areal velocity remains constant.

2.16 A particle moving in a force field has the equation of its orbit $r = a\theta$. Find the laws of force.

2.17 For which type of force, its curl is zero?

2.18 A force $\vec{F} = -C\hat{i}x$ acts on a particle. Show that $\vec{\nabla} \times \vec{F} = 0$.

In your above finding can you conclude about the nature of the force?

Numerical Problems

2.1 A particle moves in a circular orbit obeying the inverse square law. Show that for orbits of different radii, the angular momentum of the particle about the center of mass varies as the root of the radius and the total energy varies inversely as the radius.

2.2 A particle moving under a central force field described by the orbit $r = ae^{b\theta}$. Find the laws of force.

2.3 A particle moving under a central force field described by the orbit $r = a\ (1+ \cos\theta)$. Show the nature of the force graphically.

2.4 A planet revolves around a star in an elliptic orbit. The ratio of farthest distance to the closest one of the planet from the star is 2. Find the ratio of the kinetic energies of the planet at the farthest to the closest position. [*Ans:* 1: 4]

KNOW MORE

The law of conservation of energy in mechanics states that in an isolated system the total amount of energy remains constant. On the basis of this law, scientists concluded that in an isolated system, energy is neither created nor destroyed but it can change from one form to other. As an example, potential energy can be converted to kinetic energy and the kinetic energy can become thermal energy.

Activity

There is an easy to customize and extend open source Java program Central force workbench that can be used in the classroom to simulate the motion of a particle (or a two-body system) under central forces. It may be useful to illustrate problems in which the analytical solution is not available.

Interesting facts

With the formulation of the special theory of relativity Albert Einstein proposed one component of an energy-momentum 4-vector. Out of these four components, one is energy and three are momentum vector, separately conserved in any given inertial reference frame. Further, conserved is the vector length, the so called Minkowski norm, which is the rest mass. The relativistic energy of a single massive particle involves its rest mass in addition to its kinetic energy of motion. In the limit of zero kinetic energy the total energy of particle or object is associated with its rest mass**.** In the general theory of relativity, conservation of energy-momentum can be expressed with the aid of a stress-energy-momentum pseudo tensor.

Analogy

In the twentieth century, experimental observations led to a detailed knowledge of the large-scale properties of the universe. Newton's universal law of gravitation was not able to accurately model the observed universe and so there was a necessity to replace it by general theory of relativity. By the end of twentieth century and at the very beginning of the twenty first century, some new finding, *e.g.*, the accelerated expansion of the Universe have demanded introduction of new concepts like dark energy. This concept leads once again to a fundamental rethinking of the basic concepts of physics for proper interpretation of the observed phenomena.

History

Early days philosophers took the support of the conservation of some underlying substance of which everything is made. As for example, Thales of Miletus thought it was water. In 1638, Galileo published his work including the celebrated "interrupted pendulum" which can be explained using modern language as conservatively converting potential energy to kinetic energy and vice versa. But Galileo did not define the process in modern terms and so cannot be credited with the crucial insight. It was Gottfried Wilhelm Leibniz during 1676–1689 who first attempted to develop a proper mathematical formulation of the kind of energy connected with motion.

Timelines

1619: *Kepler*'s laws.

1710: *Jakob Hermann* showed that *Laplace-Runge-Lenz* vector is conserved for the inverse-square central force.

1803: *Louis Poinsot* developed idea of angular momentum conservation.

Applications (Real Life / Industrial)

Understanding the subject at depth, engineers can design machines which can be used energy to produce work. For example, thermal energy or the heat produced by burning a fuel can be utilized to drive a turbine, used to generate electricity. Based on this law, one can say that perpetual motion machines can operate only if they deliver no energy to their surroundings. However, from Albert Einstein's formulation of the special theory of relativity scientists now consider the conservation of energy as part of a larger law known as the conservation of "mass-energy".

Case Study (Environmental / Sustainability / Social / Ethical Issues)

When we are designing a control system, we need to detect the motion of the system and a large number of arrays of different sensors are available. As an example, GPS is used to determine the position on the earth's surface measuring the time for electromagnetic waves to travel from satellites in known positions in space to the sensor.

Inquisitiveness and Curiosity Topics

In 1705 Halley identified the 'Halley's Comet' with the comets of 1531 and 1607 and predicted its return in 1758, considering the general theory of motion as proposed by Newton in line with the motion of two bodies in central interaction.

Reference for Inquisitiveness and Curiosity Topics

Aguirregabiria, Juan. (2015). Simulating central forces in the classroom

REFERENCES AND SUGGESTED READINGS

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6. P. A. Tipler, *Physics for Scientists and Engineers*: *Mechanics, Oscillations and Waves, Thermodynamics*, 5th Edition, W. H. Freeman, 2004.
7. R. A. Serway and J. W. Jewett, *Physics for Scientists and Engineers*, 8th Edition, Brooks/Cole, Cengage Learning, 2010.
8. B. Stephenson, *Kepler's Physical Astronomy*, Princeton University Press, 1994.
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10. https://nptel.ac.in/courses/115/105/115105098/
11. https://www.uu.edu/dept/math/SeniorPapers/09-10/DavisEmily.pdf
12. https://web.njit.edu/~kenahn/19fall/phys102/lecture/L15-L16.pdf
13. https://web.njit.edu/~tyson/P111_chapter7.pdf
14. https://www.math.ksu.edu/~dbski/writings/planetary.pdf

3 Dynamics of Particles

UNIT SPECIFICS

We have emphasized the following aspects in this unit:

- Non-inertial frames of reference;
- Rotating coordinate system;
- Five-term acceleration formula;
- Centripetal accelerations;
- Coriolis accelerations;
- Applications: Weather systems;
- Foucault pendulum.

The applications of the topics are discussed for generating further curiosity and creativity as well as improving problem solving capacity.

Besides giving a large number of multiple choice questions as well as questions of short and long answer types marked in two categories following lower and higher order of Bloom's taxonomy, assignments through a number of numerical problems, a list of references and suggested readings are given in the unit so that one can go through them for practice. It is important to note that for getting more information on various topics of interest some QR codes have been provided in different sections which can be scanned for relevant supportive knowledge.

After the end of the unit, based on the content, there is a "Know More" section. This section has been carefully designed so that the supplementary information provided in this part becomes beneficial for the users of the book. This section mainly highlights the initial activity, examples of some interesting facts, analogy, history of the development of the subject focusing the salient observations and finding, timelines starting from the development of the concerned topics up to the recent time, applications of the subject matter for our day-to-day real life or/and industrial applications on variety of aspects, case study related to environmental, sustainability, social and ethical issues whichever applicable, and finally inquisitiveness and curiosity topics of the unit.

RATIONALE

This fundamental unit on dynamics of particles will help students to get a primary idea about the Non-inertial frames, rotating coordinate system, acceleration formula including centripetal and Coriolis accelerations. On the basis of this knowledge they will be able to understand the basics of particle dynamics. It further describes some important relations and applications which are useful for solving many fundamental problems.

Any influence that is responsible for a body to accelerate is called a force. Study of the motion of particles in terms of forces related to that motion is known as particle dynamics. A body gets a constant velocity unless it is acted on by a net force.

Dynamics of mechanical systems was first stated by Isaac Newton in his Principia and so Newton's laws form the basis for the derivation of the equations of motion for particles. The use of particle mass for representing a body is an idealized concept which provides the simplest model in dynamics. In general, dynamics is associated with the motion of material objects in relation to the physical factors which affect them.

PRE-REQUISITES

Mathematics: Differential Calculus (Class XII)

Physics: Mechanics (Class XII)

UNIT OUTCOMES

List of outcomes of this unit is as follows:

U3-O1: Describe inertial and non-inertial frame of reference

U3-O2: Explain Coriolis force and centrifugal force with expression

U3-O3: Explain the role of fictitious forces in rotating frame of references

U3-O4: Describe the effect of centrifugal force on gravity

U3-O5: Apply the concepts of Coriolis force in different natural phenomenon

Unit-3 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (*1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation*)					
	CO-1	CO-2	CO-3	CO-4	CO-5	CO-6
U3-O1	2	2	1	-	3	-
U3-O2	-	1	1	2	3	-
U3-O3	2	1	1	-	3	-
U3-O4	-	1	1	1	1	-
U3-O5	-	1	1	-	3	-

3.1 INTRODUCTION

A reference frame in which Newton's laws are valid is called inertial frame of reference. All uniformly moving reference frames with respect to inertial reference frame will also be inertial. Up to that we would study the motion of the bodies, unless otherwise studied, in inertial frames. But, when a reference frame gets acceleration with respect to inertial frame, the form of physical laws such as the Newton's second law becomes different. This type of frame of reference having acceleration relative to inertial reference frame is called as a non-inertial reference frame.

3.2 NON-INERTIAL REFERENCE FRAME

The frame of reference which is accelerated with respect to an inertial reference frame is known as the non-inertial frame of reference. Due to the acceleration in non-inertial reference frame a *pseudo force* arises. Newton's laws are no longer applicable in these reference frames. Let a reference frame S' move with a constant acceleration f_0 relative to an inertial reference frame *S*. If f and f' be the acceleration of a particle relative to S and S' frames respectively then $f' = f - f_0$

or, $$mf' = mf - mf_0$$

or, $$mf' = F - mf_0$$

or, $$f' = \frac{F - mf_0}{m} \tag{3.1}$$

which gives the acceleration of the particle in S' frame of reference.

The above equation shows that Newton's law $\left[f = \frac{F}{m}\right]$ is not valid in non-inertial frame of reference. In order to make Newton's law valid in non-inertial frame of reference an extra term $-mf_0$ is to be added with the actual force acting on the particle. This is known as the *pseudo force* or the *fictitious force*. It is the force which arises only due to the acceleration of the frame.

3.3 CENTRIPETAL AND CORIOLIS ACCELERATIONS

Rotating reference frame may be considered as a very special case of non-inertial reference frame which rotates with respect to an inertial reference frame. A very common example of a rotating frame of reference is the surface of the earth. Let us now derive expressions for velocity, acceleration and force on a particle in a uniformly rotating frame with respect to an inertial reference frame. Let, a fixed inertial frame S and a rotating non-inertial frame S' are represented by coordinates X, Y, Z and X', Y', Z'. Let, S' frame is rotating with an angular velocity $\vec{\omega}$ with respect to the S frame. Now, the magnitude of any vector $\vec{A}$ observed in the S and S' frames will be the same, though their magnitudes may differ. The positions of a particle with respect to the S and S' frames will be,

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z = \vec{r}\Big|_{fixed} \tag{3.2}$$

and $$\vec{r} = \hat{i}'x' + \hat{j}'y' + \hat{k}'z' = \vec{r}\Big|_{rotation} \tag{3.3}$$

where (x, y, z) and (x', y', z') are the positional coordinates of the particle with respect to the S and S' frames respectively. Obviously,

$$\vec{r}\Big|_{fixed} = \vec{r}\Big|_{rotation}$$

or,
$$\left.\frac{d\vec{r}}{dt}\right|_{fixed} = \left.\frac{d\vec{r}}{dt}\right|_{rotation}$$

Now,
$$\left.\frac{d\vec{r}}{dt}\right|_{fixed} = \hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt}$$

and,
$$\left.\frac{d\vec{r}}{dt}\right|_{rotation} = \hat{i}'\frac{dx'}{dt} + \hat{j}'\frac{dy'}{dt} + \hat{k}'\frac{dz'}{dt} + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}$$

or,
$$\hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt} = \hat{i}'\frac{dx'}{dt} + \hat{j}'\frac{dy'}{dt} + \hat{k}'\frac{dz'}{dt} + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}$$

or,
$$\left(\frac{d\vec{r}}{dt}\right)_S = \left(\frac{d\vec{r}}{dt}\right)_{S'} + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}$$

Now,
$$\frac{d\hat{i}'}{dt} = \vec{\omega}\times\hat{i}',\ \frac{d\hat{j}'}{dt} = \vec{\omega}\times\hat{j}' \text{ and } \frac{d\hat{k}'}{dt} = \vec{\omega}\times\hat{k}'$$

$\therefore$
$$\left(\frac{d\vec{r}}{dt}\right)_S = \left(\frac{d\vec{r}}{dt}\right)_{S'} + \vec{\omega}\times\left(x'\hat{i}' + y'\hat{j}' + z'\hat{k}'\right)$$

or,
$$\left(\frac{d\vec{r}}{dt}\right)_S = \left(\frac{d\vec{r}}{dt}\right)_{S'} + \vec{\omega}\times\vec{r} \tag{3.4}$$

or,
$$\vec{v} = \vec{v}' + \vec{\omega}\times\vec{r} = \vec{v}' + \vec{v}_0 \tag{3.5}$$

where $\vec{v}_0$ is the linear velocity due to rotation of the particle, $\vec{v}$ and $\vec{v}'$ are the velocities of the particle in S and S' frames respectively.

So, the rotation with respect to any inertial frame of reference may be represented with the help of the following operator relation:

$$\left(\frac{d}{dt}\right)_S = \left(\frac{d}{dt}\right)_{S'} + \vec{\omega}\times \tag{3.6}$$

Applying this operator relation to the velocity vector $\vec{v}$ we get

$$\left(\frac{d\vec{v}}{dt}\right)_S = \left(\frac{d\vec{v}}{dt}\right)_{S'} + \vec{\omega}\times\vec{v}$$

or, $$\left(\frac{d\vec{v}}{dt}\right)_S = \left(\frac{d\left[\vec{v}' + \vec{\omega}\times\vec{r}\right]}{dt}\right)_{S'} + \vec{\omega}\times\left(\vec{v}' + \vec{\omega}\times\vec{r}\right)$$

or, $$\left(\frac{d\vec{v}}{dt}\right)_S = \left(\frac{d\vec{v}'}{dt}\right)_{S'} + \left(\frac{d\vec{\omega}}{dt}\times\vec{r}\right)_{S'} + \vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{S'} + \vec{\omega}\times\vec{v}' + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)$$

or, $$\left(\frac{d\vec{v}}{dt}\right)_S = \left(\frac{d\vec{v}'}{dt}\right)_{S'} + \left(\frac{d\vec{\omega}}{dt}\times\vec{r}\right)_{S'} + \vec{\omega}\times\vec{v}' + \vec{\omega}\times\vec{v}' + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)$$

or, $$\left(\frac{d\vec{v}}{dt}\right)_S = \left(\frac{d\vec{v}'}{dt}\right)_{S'} + \left(\frac{d\vec{\omega}}{dt}\times\vec{r}\right)_{S'} + 2\vec{\omega}\times\vec{v}' + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)$$

or, $$\left(\frac{d\vec{v}}{dt}\right)_S = \left(\frac{d\vec{v}'}{dt}\right)_{S'} + \left(\frac{d\vec{\omega}}{dt}\times\vec{r}\right)_{S'} + 2\vec{\omega}\times\vec{v}' + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) \tag{3.7}$$

where $\vec{f}$ and $\vec{f}'$ are the accelerations of the particle in S and S' frames respectively. The term $2\vec{\omega}\times\vec{v}'$ is the *Coriolis acceleration* and the term $\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)$ represents the *centripetal acceleration* and the term $\left(\frac{d\vec{\omega}}{dt}\times\vec{r}\right)_{S'}$ is the acceleration due to the variation of angular velocity $\vec{\omega}$ with time.

If $\frac{d\vec{\omega}}{dt} = 0$ then $\vec{f} = \vec{f}' + 2\vec{\omega}\times\vec{v}' + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)$.

Again multiplying both sides of Eq. (3.7) by m we get

$$m\vec{f} = m\vec{f}' + 2m\vec{\omega}\times\vec{v}' + m\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) + m\left(\frac{d\vec{\omega}}{dt}\times\vec{r}\right)_{S'}$$

or, $$m\vec{f}' = m\vec{f} - 2m\vec{\omega}\times\vec{v}' - m\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) - m\left(\frac{d\vec{\omega}}{dt}\times\vec{r}\right)_{S'}$$

or, $$\vec{F}' = \vec{F} - 2m\vec{\omega}\times\vec{v}' - m\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) - m\left(\frac{d\vec{\omega}}{dt}\times\vec{r}\right)_{S'} = \vec{F} + \vec{F}_0 \tag{3.8}$$

where $\vec{F}'$ and $\vec{F}$ are the forces acting on the particle in the S' and S frames respectively. $\vec{F}'$ can be regarded as the effective force acting on the particle which is the sum of the true force $\vec{F}$ and the fictitious force in the non-inertial frame given by,

$$\vec{F}_0 = -2m\vec{\omega}\times\vec{v}' - m\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) - m\left(\frac{d\vec{\omega}}{dt}\times\vec{r}\right)_{S'} \tag{3.9}$$

For constant $\vec{\omega}$ we have

$$\vec{F}_0 = -2m\vec{\omega}\times\vec{v}' - m\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) = \text{Coriolis force} + \text{Centrifugal force}$$

Coriolis force is the fictitious force proportional to $\vec{\omega}$ and $\vec{v}'$, perpendicular to both $\vec{\omega}$ and $\vec{v}'$ and can act on a particle when it is in motion relative to a rotating frame.

Centrifugal force is the fictitious force that acts on a particle at rest relative to a rotating reference frame. Its magnitude is $m\omega^2 r$ and is numerically equal to the centripetal force but directed away from the axis of rotation.

3.4 CORIOLIS FORCE DUE TO EARTH'S ROTATION

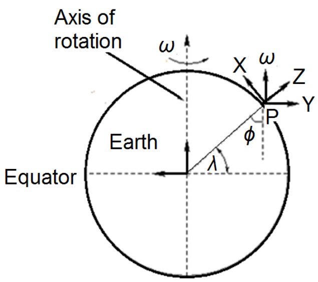


Fig. 3.1: Coriolis force due to the rotation of the Earth

In classical mechanics, Coriolis force is an inertial force proposed in 1835 by Gustave-Gaspard Coriolis. Due to the earth's rotation on its axis, air is deflected left in southern hemisphere and right in northern hemisphere. This circulating deflection of air is known as the Coriolis effect. Let a point mass m is located at a point P on the surface of the earth at a latitude λ [Fig. 3.1]. Let the inertial coordinate system (x, y, z) have its origin rigidly fixed to the earth's surface at P with the Z-axis along the vertically upward direction (along the radius of the earth). The angular velocity vector is parallel to the axis of rotation of the earth and is in the Y-Z plane given by,

$$\vec{\omega} = \omega\cos\lambda\,\hat{j} + \omega\sin\lambda\,\hat{k} \tag{3.10}$$

Now, for the horizontal projection of the particle with a velocity $\vec{v}$ we have

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} = \dot{x}\hat{i} + \dot{y}\hat{j} \qquad [\text{as } \dot{z} = 0\,] \tag{3.11}$$

Now, the Coriolis force acting on the particle will be,

$$\vec{F} = -2m\vec{\omega}\times\vec{v} = -2m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ \dot{x} & \dot{y} & 0 \end{vmatrix}$$

$$= -2m\omega\left(-\dot{y}\sin\lambda\,\hat{i} + \dot{x}\sin\lambda\,\hat{j} - \dot{x}\cos\lambda\,\hat{k}\right) \tag{3.12}$$

SCAN ME

to view the Coriolis effect on the Earth's surface

The horizontal and vertical components of this force are respectively,

$$-2m\omega\left(-\dot{y}\sin\lambda\,\hat{i} + \dot{x}\sin\lambda\,\hat{j}\,\right)$$

and $\qquad 2m\omega\dot{x}\cos\lambda\,\hat{k}$

The magnitude of the Coriolis force is thus,

$$\left|\vec{F}\right| = 2m\omega\left(\dot{y}^2\sin^2\lambda + \dot{x}^2\right)^{1/2} \tag{3.13}$$

with components along the horizontal and vertical directions as,

$$2m\omega\sin\lambda\left(\dot{x}^2 + \dot{y}^2\right)^{1/2} = 2m\omega v\sin\lambda$$

and $\quad 2m\omega\dot{x}\cos\lambda = 2m\omega v_x\cos\lambda.$

Example 3.1 A rocket is moving with a velocity of of 1.5 km/s at $\lambda = 50^0$ S. Calculate the magnitude of the Coriolis acceleration.

Solution

The Coriolis force is given by $F = 2m\omega v\cos\lambda$.

Hence the Coriolis acceleration is given by $f = 2\omega v\cos\lambda$.

Here $\lambda = 50^0$, v = 1.5 km/s and $\omega = \dfrac{2\pi}{24 \times 3600}$ rad/s.

So, $\quad f = 2\omega v\cos\lambda = 2 \times \dfrac{2\pi}{24 \times 3600} \times 1500 \times \cos 50 = 0.14\,\text{m/s}^2.$

EXAMPLE 3.1

3.5 CORIOLIS FORCE ON A FREELY FALLING BODY

On the earth's surface, let a point mass m is located at a point P at a latitude λ and altitude h [Fig. 3.1]. We assume that the inertial coordinate system (x, y, z) have its origin rigidly fixed to the earth's surface at P with the Z-axis along the vertically upward direction (along the radius of the earth).

The angular velocity vector is parallel to the axis of rotation of the earth and is in the Y-Z plane given by,

$$\vec{\omega} = \omega\cos\lambda\hat{j} + \omega\sin\lambda\hat{k}$$

Now, for the vertical fall of the particle with a velocity $\vec{v}$ we have

$$\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} = \dot{z}\hat{k} \qquad [\because \dot{x} = 0, \dot{y} = 0]$$

Now, the Coriolis force acting on the particle will be,

$$\vec{F} = -2m\vec{\omega} \times \vec{v} = -2m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ 0 & 0 & \dot{z} \end{vmatrix}$$

$$= -2m\omega\dot{z}\cos\lambda\hat{i} = 2m\omega v\cos\lambda\hat{i} \tag{3.14}$$

since the body is falling vertically downward, $\dot{z} = -v$.

The Coriolis force will therefore be along the positive X-axis acting towards east in the northern hemisphere.

The magnitude of the acceleration on the particle due to this force will be,

$$\ddot{x} = 2\omega v\cos\lambda = 2\omega g t\cos\lambda$$

Integrating twice with respect to t we get,

$$\dot{x} = \omega g t^2 \cos\lambda \qquad [\text{as } \dot{x} = 0 \text{ during the start of the motion}] \tag{3.15}$$

and

$$x = \frac{1}{3}\omega g t^3 \cos\lambda \qquad [\text{as } x = 0 \text{ during the start of the motion}] \tag{3.16}$$

This will be the displacement of the body at a time t along the positive X-axis towards east.

Now for a freely falling body from a height h we have, $h = \frac{1}{2}gt^2$

or,

$$t = \sqrt{\frac{2h}{g}}$$

$\therefore$

$$x = \frac{1}{3}\omega g\left(\sqrt{\frac{2h}{g}}\right)^3 \cos\lambda = \frac{2}{3}\omega h\sqrt{\frac{2h}{g}}\cos\lambda \tag{3.17}$$

This gives the displacement of the body at a time t along the positive X-axis towards east in the northern hemisphere in terms of h.

EXAMPLE 3.2

Example 3.2 Find the deflection of a freely falling body from a height of 50 m at 30^0 N latitude.

Solution

The deflection is given by

$$x = \frac{1}{3}\omega g\left(\sqrt{\frac{2h}{g}}\right)^3 \cos\lambda.$$

Here, h = 50 m and λ= 30^0 N.

So,

$$x = \frac{1}{3}\omega g\left(\sqrt{\frac{2h}{g}}\right)^3 \cos\lambda = \frac{1}{3}\times\frac{2\pi\times 981}{24\times 60\times 60}\left(\sqrt{\frac{2\times 5000}{981}}\right)^3 \cos 30$$

$$= 0.67 \text{ cm.}$$

3.6 EFFECT OF CENTRIFUGAL FORCE ON GRAVITY

Centrifugal force is an inertial force acts on all objects when observed in a rotating reference frame. This force is moving away from the axis parallel to the rotation axis and going through the origin of the coordinate system. When the rotation axis goes through the origin, the centrifugal force moves radially outwards from the axis.

Let the earth is rotating with angular velocity $\vec{\omega}$ so that the reference frame fixed to it is a rotating frame. Let, a particle of mass m is located at a point P (with latitude λ) and is at rest [Fig. 3.2].

So, there will not be any Coriolis acceleration acting on the particle and only the effect of centrifugal acceleration is there. Now, if $\vec{\omega}$ is acting along the Y-axis then we have,

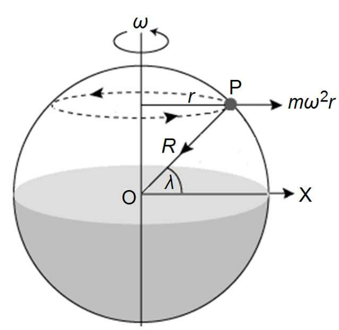


Fig. 3.2: Effect of centrifugal force on gravity due to the rotation of the Earth

$$\vec{\omega} = \omega\hat{j} \tag{3.18}$$

Now, the components of the acceleration due to gravity (g) along the X and Y directions will be,

$$-\mathrm{g}\cos\lambda\hat{i} \text{ and } -\mathrm{g}\sin\lambda\hat{j}$$

Such that $\vec{g} = -\mathrm{g}\cos\lambda\hat{i} - \mathrm{g}\sin\lambda\hat{j}$

Now, the observed value of acceleration due to gravity (g) at P with latitude λ due to the centrifugal force will be,

$$\vec{g}_\lambda = \vec{g} - \vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right) \tag{3.19}$$

where $\vec{r}$ is the distance of the point P with respect to the axis of rotation.

$\therefore$ $\vec{r} = R\cos\lambda\hat{i}$ [R being the radius of the earth]

So, $\vec{g}_\lambda = -\mathrm{g}\cos\lambda\hat{i} - \mathrm{g}\sin\lambda\hat{j} - \omega\hat{j}\times\left(\omega\hat{j}\times R\cos\lambda\hat{i}\right) = -\left[\left(\mathrm{g}-\omega^2R\right)\cos\lambda\hat{i} + \mathrm{g}\sin\lambda\hat{j}\right]$

or, $g_\lambda = \left[\left(\mathrm{g}-\omega^2R\right)^2\cos^2\lambda + \mathrm{g}^2\sin^2\lambda\right]^{1/2} = \mathrm{g}\left[\left(1-\dfrac{2\omega^2R}{g}\right)\cos^2\lambda + \sin^2\lambda\right]^{1/2} \left[\because \dfrac{\omega^2R}{g} \ll 1\right]$

or, $$g_\lambda \simeq \mathrm{g}\left[1-\frac{\omega^2R}{g}\cos^2\lambda\right] = g - \omega^2R\cos^2\lambda \tag{3.20}$$

Thus due to the rotation of the earth, the value of g at a latitude λ will be decreased by $\omega^2R\cos^2\lambda$.

At equator, $\lambda = 0^{\circ}$ and hence $g_{equator} = g - \omega^2R$ (3.21)

At poles, $\lambda = 90^{\circ}$ and hence $g_{pole} = g$ (3.22)

So, the effect of centrifugal force due to the earth's rotation is maximum at the equator and nil at the poles and the difference is,

$$\omega^2R = \left(\frac{2\pi}{24\times60\times60}\right)^2\times6400000 = 0.0337\ \mathrm{m/s^2}.$$

Example 3.3 If the earth were to cease rotating about the axis, what will be the change in the value of g at a place having $\lambda = 50^0$? (assume earth's radius = 6400 km)

Solution

Here $\lambda = 50^0$, R= 6400 km = 6400000 m and $\omega = \dfrac{2\pi}{24\times3600}$ rad/s.

Using, $g_\lambda = g - \omega^2R\cos^2\lambda$

we get, $g - g_\lambda = \omega^2R\cos^2\lambda = \left(\dfrac{2\pi}{24\times3600}\right)^2\times6400000\times\cos^2 50$

$= 1.4\,\mathrm{cm/s^2}$.

EXAMPLE 3.3

3.7 APPLICATIONS OF CORIOLIS FORCE

Though the relative magnitude of the Coriolis acceleration is small as compared to the real acceleration but it plays an important role in many terrestrial phenomena. Some of the geographical effects of Coriolis force due to the rotation of the earth are the deflection in the rotatory effect in cyclones, direction of trade winds, erosion of the right bank of river *etc*.

Rotatory effects in cyclones: In the atmosphere presumably due to different heating of air a low pressure region develops. Such developed low pressure regions, with roughly concentric isobars are known as the cyclones. If in the northern hemisphere, a low pressure region develops the air from all other directions of the surrounding region rushes towards that low pressure region at right angles to the isobars.

This rushing air is then deflected towards the right due the Coriolis force and this causes an anti-clockwise circulation (cyclone) of the air around the low pressure region. This type of circulation goes on till the thrust due to pressure gradient is balanced by that due to Coriolis force. In the southern hemisphere the direction of the cyclone is clockwise. At the equator ($\lambda = 0$), the horizontal component of Coriolis force is $2m\omega v\sin\lambda$ which effectively causes a zero deflection of the wind. That is why no cyclone occurs on the equator.

Deflection of trade wind: As the earth's surface at the equator gets heated up by Sun rays, the air in contact is also heated and rises up. But the rushing air does not follow the north south direction in the northern hemisphere. Due to the Coriolis force, it gets deflected towards the west. The angular velocity vector in the expression for the Coriolis force [$-2m\vec{\omega} \times \vec{v}$] is directed along the axis of the earth upward from the north pole.

The wind flows from the north to south, tangential to the surface of the earth and there is a velocity component perpendicular to the axis and directed away from it. By using vector algebra, we can say that this force will deflect the moving particles to their right *i.e.*, the particles will be deflected towards the west direction in the northern hemisphere giving rise to the north-west trade wind. In a similar way, the rushing air from the south will be deflected to the right *i.e.*, the particles will be deflected to the east direction in southern hemisphere giving rise to the south-east trade wind.

Erosion of right bank of rivers: The water of rivers during their course from north to south (or from south to north) experience Coriolis force towards its right due to the rotation of the earth. Thus the right bank of river gets eroded more rapidly as compared to the left bank. In a similar manner, the right bank is also getting steeper than the left bank.

In the northern hemisphere for a river flowing from north to south, the right bank is to the west and for a river flowing from south to north; the right bank is to the east. The effect of Coriolis acceleration during the flight of a guided missile assumes considerable significance as the velocity and the time of flight are considerably large.

3.8 FOUCAULT PENDULUM

It is a pendulum moves under the influence of gravity, suspended from a long cable with tension T. It can detect even the slow rotation of the earth. The equation of motion is

$$\vec{T} - mg\hat{k} - 2m\vec{\omega} \times \vec{v} = m\vec{f} \tag{3.23}$$

where the centrifugal force makes only a slight change in the magnitude and direction of g,

$$\left|\vec{\omega}\times\left(\vec{\omega}\times\vec{v}\right)\right| = \omega^2 r\cos\alpha \tag{3.24}$$

with direction away from the axis of rotation. Let the angle of oscillation of the pendulum be very small (and the supporting cable very long) so that the arc of the pendulum is sufficiently close to horizontal motion. The net external force is the combined effect of the tension and gravity, equal to a restoring force, $mg\sin\theta$, so

$$mg\sin\theta \approx mg\theta = \frac{mg}{L}L\theta$$

giving an effective Hooke's law force with spring constant mg/L. The restoring force is therefore

$$\vec{T} - mg\hat{k} \simeq \frac{mg}{L}\rho\left(\hat{i}\cos\varphi + \hat{j}\sin\varphi\right) \tag{3.25}$$

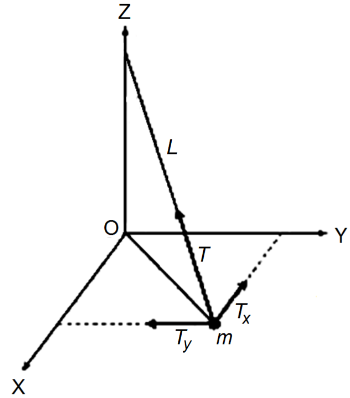


Fig. 3.3: Foucault pendulum

where $$\rho = \sqrt{x^2 + y^2} \tag{3.26}$$

and motion in the $\hat{k}$ direction is negligible. The velocity and angular velocity at latitude λ are

$$\vec{v} = \hat{i}\dot{x} + \hat{j}\dot{y}$$

$$\vec{\omega} = \omega\cos\lambda\hat{j} + \omega\sin\lambda\hat{k}$$

and the equation of motion becomes

$$m\left(\hat{i}\ddot{x} + \hat{j}\ddot{y} + \hat{k}\ddot{z}\right) = -\frac{mg}{L}\rho\left(\hat{i}\cos\varphi + \hat{j}\sin\varphi\right) - 2m\omega\left(\cos\lambda\hat{j} + \sin\lambda\hat{k}\right)\times\left(\hat{i}\dot{x} + \hat{j}\dot{y}\right)$$

or, $$\hat{i}\ddot{x} + \hat{j}\ddot{y} + \hat{k}\ddot{z} = -\frac{g}{L}x\hat{i} - \frac{g}{L}y\hat{j} + 2\omega\dot{x}\hat{k}\cos\lambda - 2\omega\dot{x}\hat{j}\sin\lambda + 2\omega\dot{y}\hat{i}\sin\lambda$$

Equating like components we get,

$$\ddot{x} = -\frac{g}{L}x + 2\omega\sin\lambda\dot{y} \tag{3.27 (a)}$$

$$\ddot{y} = -\frac{g}{L}y - 2\omega\sin\lambda\dot{x} \tag{3.27 (b)}$$

and $$\ddot{z} = 2\omega\cos\lambda\dot{x} \tag{3.27 (c)}$$

The Z-direction may be ignored since any acceleration in this direction acts only as a mild perturbation on the acceleration of gravity [$\omega\dot{x} << g$.]. Then we have

$$\ddot{x} + \frac{g}{L}x = 2\omega\sin\lambda\dot{y} \tag{3.28 (a)}$$

$$\ddot{y} + \frac{g}{L}y = -2\omega\sin\lambda\dot{x} \tag{3.28 (b)}$$

So, the equations of motion will be,

$$m\ddot{x} = -\frac{mg}{L}x + 2m\omega\sin\lambda\dot{y}$$

$$m\ddot{y} = -\frac{mg}{L}y - 2m\omega\sin\lambda\dot{x}$$

Hence, the angular velocity of oscillation of the pendulum is, $\omega_0 = \sqrt{\frac{g}{L}}$

and thus the period of the pendulum is $\tau = 2\pi\sqrt{\frac{L}{g}}$.

Now Eq. (3.28) can be re-written as,

$$\ddot{x} = -\omega_0^2 x + 2\omega\sin\lambda\dot{y}$$

and $$\ddot{y} = -\omega_0^2 y - 2\omega\sin\lambda\dot{x}$$

Now let, $$\vec{u} = x + iy \tag{3.29}$$

so that $$\ddot{\vec{u}} + 2\omega i\sin\lambda\,\dot{\vec{u}} + \omega_0^2\vec{u} = 0$$

or, $$\ddot{\vec{u}} + 2i\,\omega_z\,\dot{\vec{u}} + \omega_0^2\vec{u} = 0$$

or, $$\left(D^2 + 2i\,\omega_z D + \omega_0^2\right)\vec{u} = 0$$

or, $$D^2 + 2i\,\omega_z D + \omega_0^2 = 0 \qquad [\text{As, } \vec{u} \neq 0]$$

$\therefore$ $$D = -i\left(\omega_z \mp \sqrt{\omega_z^2 + \omega_0^2}\right) = -i\omega_z \pm i\omega_1$$

where, $$\omega_1^2 = \omega_z^2 + \omega_0^2$$

So, the general solution of Eq. (3.29) will be,

$$\vec{u} = Ae^{-i(\omega_z - \omega_1)t} + Be^{-i(\omega_z + \omega_1)t} = e^{-i\omega_z t}\left(Ae^{i\,\omega_1 t} + Be^{-i\,\omega_1 t}\right) \tag{3.30}$$

where A and B are two arbitrary constants.

As, $\omega_z << \omega_0$ we can write,

$$\omega_1^2 = \omega_0^2 \Rightarrow \omega_1 = \omega_0$$

Thus, $$\vec{u} = Ae^{i\,\omega_0 t} + Be^{-i\,\omega_0 t} \tag{3.31}$$

which effectively describes the elliptical trajectory traced out by the bob of the pendulum. The factor $e^{-i\omega_z t}$ in Eq. (3.30) imparts a rotation on the ellipse about the vertical Z-axis with an angular velocity,

$$\omega_z = \omega\sin\lambda$$

So, the period of rotation about the vertical axis will be,

$$T = \frac{2\pi}{\omega_z} = \frac{2\pi}{\omega\sin\lambda} \tag{3.32}$$

At the equator $\lambda = 0^\circ$, hence T is infinite.

At the pole $\lambda = 90^\circ$, hence $T = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{2\pi}{24}} = 24$ h.

SCAN ME

for more on Foucault pendulum

This means that the plane of oscillation at the poles makes complete revolution in 24 h. For any other λ, $T > 24$ h. When viewed from the top, in the northern hemisphere, the rotation of the plane of oscillation is clockwise while in the southern hemisphere it is anti-clockwise. This relative angular shift of the plane of oscillation directly emphasizes the rotation of the earth about its own axis. So, we cannot strictly consider the earth as an inertial reference frame.

Example 3.4 Find the rotational rate of the plane of oscillation of a pendulum at $\lambda = 50^0$ and hence calculate the time taken by it when it is turned at right angle.

Solution

The period of rotation will be,

$$T = \frac{2\pi}{\omega \sin\lambda} = \frac{2\pi}{\frac{2\pi}{24} \times \sin 45} = 24\sqrt{2}\text{ h.}$$

So, the rate of rotation of the plane of oscillation will be,

$$\Omega = \frac{2\pi}{24\sqrt{2}}\text{ rad/h} = \frac{\pi}{12\sqrt{2}}\text{ rad/h.}$$

and the time taken by it to turn through a right angle is,

$$t = \frac{\pi/2}{\pi/12\sqrt{2}} = 6\sqrt{2}\text{ h.}$$

EXAMPLE 3.4

Example 3.5 Foucault's pendulum oscillates along north-south direction where the latitude is 30^0. What time will be elapsed before the pendulum starts to oscillate along northeast-southwest direction?

Solution

Period $$T = \frac{2\pi}{\omega \sin\lambda} = \frac{2\pi}{\frac{2\pi}{24} \times \sin 30} = 48\text{h.}$$

So, the rate of rotation of the plane of oscillation will be,

$$\Omega = \frac{2\pi}{48}\text{ rad/h} = \frac{\pi}{24}\text{ rad/h.}$$

and the time taken by the pendulum to start oscillate along northeast-southwest direction $$t = \frac{\pi/4}{\pi/24} = 6\text{h.}$$

EXAMPLE 3.5

UNIT SUMMARY

- ***Non-inertial frames of reference***

$$f' = \frac{F - mf_0}{m}$$

- ***Rotating frame of reference: Centripetal and Coriolis accelerations***

$$\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r} = \vec{v}' + \vec{v}_0$$

$$\vec{f} = \vec{f}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right) + \left(\frac{d\vec{\omega}}{dt} \times \vec{r}\right)_{S'}$$

Coriolis force

$$\vec{F}_0 = -2m\vec{\omega} \times \vec{v}' - m\vec{\omega} \times \left(\vec{\omega} \times \vec{r}\right)$$

= *Coriolis force* + *Centrifugal force*

- ***Coriolis force due to the rotation of the earth***

$$\vec{F} = -2m\vec{\omega} \times \vec{v}$$

$$= -2m\omega\left(-\dot{y}\sin\lambda\hat{i} + \dot{x}\sin\lambda\hat{j} - \dot{x}\cos\lambda\hat{k}\right)$$

$$\left|\vec{F}\right| = 2m\omega\left(\dot{y}^2\sin^2\lambda + \dot{x}^2\right)^{1/2}$$

- ***Coriolis force acting on a freely falling body***

$$\vec{F} = -2m\vec{\omega} \times \vec{v}$$

$$= -2m\omega\dot{z}\cos\lambda\hat{i} = 2m\omega v\cos\lambda\hat{i}$$

$$t = \sqrt{\frac{2h}{g}}$$

$$x = \frac{1}{3}\omega g\left(\sqrt{\frac{2h}{g}}\right)^3 \cos\lambda$$

- ***Effect of centrifugal force on gravity due to the rotation of the earth***

$$g_\lambda = g - \omega^2 R\cos^2\lambda$$

- ***Foucault pendulum***

$$\omega_0 = \sqrt{\frac{g}{L}}$$

$$T = \frac{2\pi}{\omega_z} = \frac{2\pi}{\omega\sin\lambda}$$

EXERCISES

Multiple Choice Questions

3.1 All the frames of reference that are rotating relative to a fixed frame of reference are the ___________ frame of reference

(a) inertial (b) non-inertial (c) real (d) imaginary

3.2 In a cyclone the wind whirls in the _________ sense in the southern hemisphere

(a) upwards (b) downwards (c) clockwise (d) anticlockwise

3.3 A slider sliding at 10 cm/s on a link which is rotating at 60 r.p.m. is subjected to Coriolis acceleration of magnitude

(a) $40\pi^2$ cm/s^2 (b) 0.4π cm/s^2 (c) 40π cm/s^2 (d) 4π cm/s^2

3.4 A body in motion will be subjected to Coriolis acceleration when that body is

(a) in plane rotation with variable velocity (b) in plane translation with variable velocity

(c) in plane motion which is a resultant of plane translation and rotation

(d) restrained to rotate while sliding over another body

3.5 The Coriolis effect is caused by

(a) gravity (b) the sun's radiation (c) the rotation of the earth

(d) the revolution of the earth around the sun

3.6 Because of the Coriolis effect, winds and other moving fluids are deflected to which direction in the southern hemisphere?

(a) right (b) east (c) left (d) west

3.7 The Foucault pendulum provides evidence that the earth

(a) rotates on its axis (b) has a spherical shape (c) is inclined in its axis

(d) revolves around the sun

3.8 The best evidence that the earth rotates is provided by

(a) a freely swinging Foucault pendulum (b) the different phases of moon

(c) the rising and setting of stars (d) the changing season

Answers of Multiple Choice Questions
3.1 (b), 3.2 (c), 3.3 (c), 3.4 (d), 3.5 (c), 3.6 (c), 3.7 (a), 3.8 (a)

Short and Long Answer Type Questions

Category I

3.1 Calculate the total force and fictitious force acting on a body in a non-inertial reference. Give example.

3.2 Derive the expression for the force in a uniformly rotating frame of reference.

3.3 Describe the construction of a Foucault's pendulum. Show that the bob of the pendulum describes an ellipse that rotates with an angular velocity $\omega\sin\lambda$, where the symbols are of their usual significance.

3.4 Give some nature examples what happen due to the Coriolis force.

3.5 What do you mean by a real force? What is the difference of real force with fictitious force? Cite some example to explain your answer.

3.6 What do you mean by inertial reference frame? What is the difference of inertial reference frame with non-inertial reference frame? Explain with examples.

3.7 A reference frame rotates with respect to another reference frame with an angular velocity $\vec{\omega}$. Calculate the acceleration of a particle in the rotating frame. Explain the significance of each term in the expression.

3.8 Assuming the earth to be a rotating reference frame, explain the role of centrifugal force to change the value of g. Show that g is maximum at the equator and minimum at the poles.

3.9 Calculate the vertical deviation suffered by a body while dropped from rest from a given height due to Coriolis force. Will there be any change in this deviation in the northern and the southern hemispheres?

Category II

3.10 Write down the mathematical expressions for Coriolis force and centrifugal force. When a merry-go-round is just started, a force other than the Coriolis force and centrifugal force is experienced. What is the force due to? Write down its mathematical expression.

3.11 What will be the direction of the realized Coriolis in the northern and the southern hemispheres?

3.12 Why no cyclones occur at the equator?

3.13 Why the right banks of rivers are found to be eroded more than the left bank in both the hemispheres?

3.14 Under what conditions is the Coriolis acceleration present? Discuss the geographical effect of Coriolis force due to the earth's rotation.

3.15 What is Coriolis force? Under which circumstances it is zero? When it can be maximum?

Numerical Problems

3.1 Calculate the time required for the plane of vibration of a Foucault's pendulum to rotate once at latitude of 45^0. How much time will it take to rotate to 60^0 at the same place and at a place of latitude 60^0. [*Ans:* $24\sqrt{2}$h, $4\sqrt{2}$h, $8/\sqrt{3}$h]

3.2 Find the rate of rotation of the plane of oscillation of a Foucault's pendulum at latitude of 30^0 and hence calculate the time it will take to turn through 90^0. [*Ans:* $\pi/24$ rad/h, 12h]

3.3 A particle is thrown downward with an initial speed v_0. Show that after a time t it is deflected east of the vertical by an amount

$$\omega v_0 \sin\lambda t + \frac{1}{2}\omega g t^2 \sin\lambda .$$

3.4 Find the deviation suffered by a body dropped from a height of 100 m at 60^0 N, at the equator and at the north pole. [*Ans:* 0.011, 0.022, 0]

KNOW MORE

A 'Particle' is considered as a point mass at some point in space. A particle can move about but has no characteristic orientation but is characterized by its mass. The dynamics of mechanical systems was first pointed out by Isaac Newton in his Principia of 1687. Use of particle mass for representing a body is an idealized concept which provides the simplest model in dynamics. When the laws of Newton are to be applied for a rotating reference frame, there should be an inclusion of an inertial force. For clockwise rotation it should act left of the motion of the body and for anti-clockwise rotation it should act right of the motion of the body.

Activity

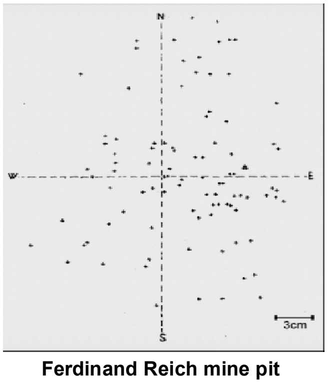
In physics, the Newtonian dynamics can be well understood as the dynamics of a particle or as a small body according to Newton's laws of motion. Study of dynamics can be divided into categories; linear dynamics and rotational dynamics. Linear dynamics are associated with the objects moving in a line and covers topics like force, mass, inertia, displacement, velocity, acceleration and momentum.

Rotational dynamics, on the other hand, are rotating or moving in a curved path and covers topics like torque, moment of inertia, rotational inertia, angular displacement, angular velocity, angular acceleration and angular momentum.

Interesting facts

In 1668 the first experimental study for the demonstration of the Coriolis force was done by Giovanni Borelli, when he considered the falling body problem on the rotating Earth. In his theoretical observation, he found that during their fall the bodies will undergo a little deflection along the east.

During the same time, more experiments were carried out on freely falling bodies. One such famous experiment was conducted by Ferdinand Reich in a mine pit. On a total of 106 experiments he noticed an average 2.8 cm deflection along the east in agreement with the theoretical value.



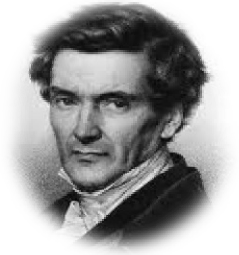
Ferdinand Reich mine pit experimental plot

Analogy

Dynamics is a very important branch of physical science and subdivision of mechanics which is associated with the motion of material objects in relation to various physical factors which affect them, *e.g.*, force, mass, momentum, energy. In fact, the dynamics of a particle involve the forces acting on the particle which is resulted in an acceleration of the particle. Study of the dynamics of a particle is also reasonably referred to as kinetics.

Foucault had derived a simple equation, known as his sine law. This gives the time it would take for a pendulum at any given latitude for completing a rotation. At the equator, the pendulum's plane of oscillation would never move but at the north pole, the plane of the pendulum would complete a 360 degree rotation in 24 hours. In Paris, this pendulum would turn 270 degrees in a day.

History


Gustave-Gaspard Coriolis

Newton developed the fundamental physical laws which help to govern dynamics in physics. By getting a knowledge Newton developed on different systems of mechanics, one can clearly get an idea of dynamics. In fact, dynamics is widely related to Newton's second law of motion. But it is true that all the three laws of motion are to be taken into account as these are interrelated in any given experiment or observation.

The work of Gustave-Gaspard Coriolis, in 1835 on the derivation for an expression of the force acting in a rotating system was revolutionarily inspired by rotating devices like industrial waterwheels.

However, the Earth is one of the rotating devices that have always been with us. Perhaps, well before Coriolis all the earliest studies to investigate the behavior of moving objects in a rotating system were realized by encompassing the diurnal rotation of the Earth.

Timelines

1668: First detailed study on a manifestation of the 'Coriolis' force by *Giovanni Borelli*.

1749: Coriolis acceleration equation was derived by *Leonhard Euler*.

1778: *Pierre-Simon Laplace* described the effect of Coriolis acceleration in tidal equations.

1798: *Laplace*'s work on the deflection of projectiles.

1831: Results of 106 experiments on freely falling balls in a mine pit by *Ferdinand Reich*.

1835: Development of the expression of force acting in rotating system by *Gustave-Gaspard Coriolis*.

1851: *Léon Foucault* showed the Earth's rotation with a huge pendulum, the well known Foucault pendulum.

1856: *William Ferrel* adopted *Laplace*'s tidal theory and introduced them in meteorology.

Applications (Real Life / Industrial)

Coriolis effect has great impact in astrophysics and stellar dynamical properties. It is indeed a controlling factor in the directions of rotation of the sunspots. It has also significant impact in several aspects of meteorology and oceanography where the motions over the surface of the Earth, which is considered to be a rotating reference frame, are subject to acceleration.

In electrodynamical studies of the generated instantaneous voltage in a rotating electrical machine can be divided into an induced voltage and a generated voltage. This is basically a relative consequence and depends on the choice of reference frames. Depending on the relative velocity of the frames it can be entirely or partially induced and or generated. Due to the rotation of the reference frame with a different velocity from that of the rotor there is a developed voltage analogous to the Coriolis force and is referred as the *Christoffel voltage*.

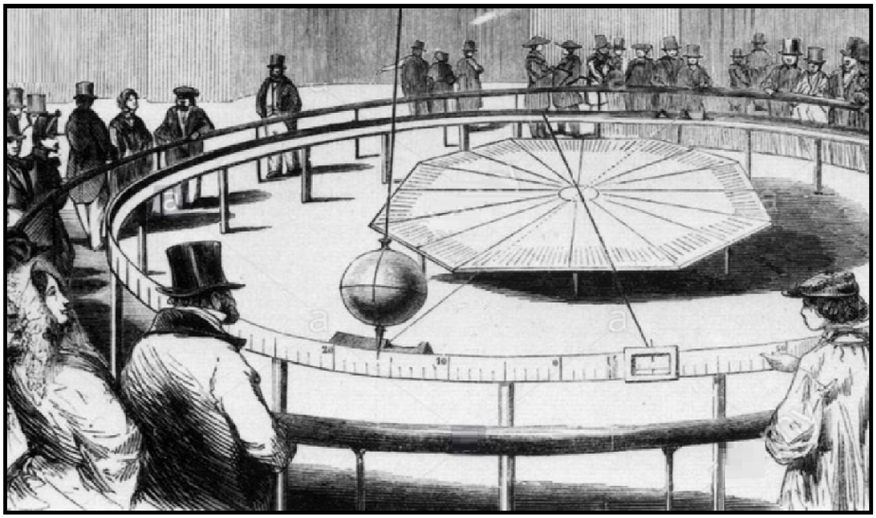
Case Study (Environmental / Sustainability / Social / Ethical Issues)

Generally speaking, scientists involved in the study of dynamics concentrate mainly on how a physical system might be developed or can be altered over time. One priority area of the researchers is to study the causes of the changes of the physical systems. The Coriolis force appears significantly in studies of the atmospheric dynamics, hydrosphere and ballistics. In case of atmospheric dynamics it affects prevailing air circulation and the rotation of storms, in hydrosphere it affects the rotation of the oceanic currents and in ballistics it can specifically effect in the launching and orbiting of space vehicles.

Inquisitiveness and Curiosity Topics

One night at about 2 AM in early January 1851, Foucault realized that if he could devise a way to hang a pendulum from the ceiling in such a manner that the pendulum was free to swing in any direction, he would be able to observe the effect of Earth's rotation. It would appear that the pendulum's path was shifting slowly but the pendulum's plane of oscillation would stay fixed while Earth turned beneath it. He realized the pendulum had to be designed very carefully wherein the bob must be perfectly symmetrical.

When starting the pendulum swinging, it had to be released gently as the slightest push would ruin the demonstration. It was the first clear and dramatic demonstration of the Earth's rotation by him. After successfully completing the experiment in his basement, he was ready to demonstrate the same on a larger scale. On February 2, 1851, Foucault sent a notice to the scientists in Paris, saying "You are invited to see the Earth turn."



Engraving of Leon Foucault's pendulum which demonstrated the Earth's rotation in the Pantheon, Paris in 1851 (Courtesy: Photography by Pictorial Press)

The next day, in the Meridian Room of the Paris Observatory, the scientists assembled there did indeed witness the Earth turn and thus the first pendulum demonstration was a success. Napoleon III, who was also an amateur scientist and supportive of Foucault, arranged for him to hold the position of Physicist attached to the Imperial Observatory, where he made significant improvements to the telescopes. But the French Academy of Science was reluctant to elect Foucault the membership. However after applying several times, Foucault was finally elected in 1865 but he died in Paris on February 11, 1868, only at age 49. The public continues to be fascinated with the Foucault pendulums, which can be seen in science museums and at many other public spaces around the world.

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4 Oscillations

UNIT SPECIFICS

We have discussed the following topics in this unit:

- Characteristics of simple harmonic motion;
- Solution of the differential equation of motion of simple harmonic motion;
- Harmonic oscillator;
- Damped harmonic motion;
- Critically damped and lightly-damped oscillators;
- Logarithmic decrement, relaxation time and quality factor associated with damped harmonic motion;
- Forced oscillations and
- Resonance in forced vibrations.

The practical applications of the topics are discussed for generating further curiosity and creativity as well as improving problem solving capacity.

Besides giving a large number of multiple choice questions as well as questions of short and long answer types marked in two categories following lower and higher order of Bloom's taxonomy, assignments through a number of numerical problems, a list of references and suggested readings are given in the unit so that one can go through them for practice. It is important to note that for getting more information on various topics of interest some QR codes have been provided in different sections which can be scanned for relevant supportive knowledge.

After the related practical, based on the content, there is a "Know More" section. This section has been carefully designed so that the supplementary information provided in this part becomes beneficial for the users of the book. This section mainly highlights the initial activity, examples of some interesting facts, analogy, history of the development of the subject focusing the salient observations and finding, timelines starting from the development of the concerned topics up to the recent time, applications of the subject matter for our day-to-day real life or/and industrial applications on variety of aspects, case study related to environmental, sustainability, social and ethical issues whichever applicable, and finally inquisitiveness and curiosity topics of the unit.

RATIONALE

This unit on oscillations will help students to get a clear idea about harmonic oscillator including damped harmonic motion and over-damped condition. It will help to develop some important ideas on oscillatory motion. The explanation on forced oscillations and resonance are topics whose understandings help in different applied areas. Some related phenomena are also outlined as supporting topics with large area of applications.

Oscillation generally means the repeated to-and-fro motion of some objects between two positions or states. A single oscillation is considered as a complete movement, whether up and down or side to side, over a period of time. In broadest sense, oscillation can occur in anything from a person's decision-making process to tides and the pendulum of a clock. For example, in a pendulum-driven clock, the oscillation is the back and forth movement of the pendulum. Waves can also be considered as oscillations, or vibrations, about a rest position. Sound wave causes air particles to vibrate back and forth. Ripples cause water particles to vibrate up and down.

PRE-REQUISITES

Mathematics: Trignometry (Class XII)

Physics: Simple harmonic Motion, Waves (Class XII)

UNIT OUTCOMES

List of outcomes of this unit is as follows:

U4-O1: Describe the relationship between displacement, velocity and acceleration for a mass moving in simple harmonic manner

U4-O2: Explain different types of damped vibratory motion with the idea of damping constant

U4-O3: Explain steady states and variable state in forced vibratory motion

U4-O4: Discuss velocity, energy resonance and sharpness of resonance

U4-O5: Apply differential equation of simple harmonic motion to calculate time period of an oscillatory motion

Unit-4 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (*1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation*)					
	CO-1	CO-2	CO-3	CO-4	CO-5	CO-6
U4-O1	-	-	2	-	-	3
U4-O2	-	-	1	-	-	3
U4-O3	-	-	1	-	-	3
U4-O4	-	-	1	-	-	3
U4-O5	-	-	1	-	-	3

4.1 INTRODUCTION

The repeated nature of any motion at a regular interval of time is known as *periodic motion*. Motion of a pendulum, oscillation of an object on a spring, earth's diurnal motion *etc* are examples of such motion. If a particle in course of its periodic motion follows to-and-fro movement about equilibrium or the mean position, then this type of motion is named as *oscillatory motion.* Motion of a simple pendulum and those of a spring are examples of such motion. All the oscillatory motions are periodic but all periodic motions are not oscillatory.

Number of oscillation made by an oscillating body per unit time is termed as frequency (n) and the time required to complete one oscillation is called the time period (T) of the motion. The relationship between these two is given by:

$$n = \frac{1}{T} \tag{4.1}$$

When a body performs steady vibration with its natural angular frequency, it experiences a number of dissipative forces or frictional forces which decreases the amplitude of vibration of the body with time and ultimately the motion of body is stopped. Such vibration of the body with decaying amplitude is called *damped vibration*. The amplitude of oscillation of a body undergoing damped vibration goes on decreasing with time due to loss of energy to overcome the resistive or dissipative forces. When an external periodic force is applied to the system to make up for the losses, the amplitude of vibration does not decay with time and the body vibrates regularly under the action of the periodic force. This vibration of the body under the action of the periodic forces is termed as the *forced vibration*.

4.2 SIMPLE HARMONIC MOTION

In both physics and mechanics, simple harmonic motion (SHM) is considered as a special case of periodic motion. In this type of motion, the restoring force acting on any moving object is directly proportional to the magnitude of the displacement of the object and acts towards the equilibrium position of the object

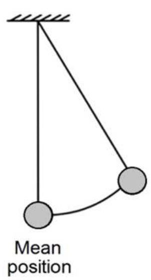

Fig. 4.1: Motion of a simple pendulum

When a particle moves to-and-fro in such a way that the acceleration of it is always directed towards a fixed point in its path of motion and is proportional to the displacement from the mean position, the motion of a particle is termed as SHM, *e.g.*, motion of a simple pendulum [Fig. 4.1]. The distance traversed by a particle at any instant from the mean position is termed as the displacement of the particle and the maximum displacement on either side from the mean position is the amplitude of the motion.

If a particle of mass m executes SHM about a mean position such that at any time t the displacement from the mean position is x then the acceleration of the particle can be written as:

$$f = -kx \tag{4.2}$$

Here k is a constant of proportionality.

So the force experienced by the particle at that instant is given by,

$$F = mf = -mkx = -\mu x \tag{4.3}$$

μ is the proportionality constant and is termed as the resorting force per unit displacement of the particle.

4.2.1 Characteristics of SHM

The characteristics of SHM are the following:

(a) The motion is oscillatory,

(b) The motion is linear, and

(c) From the mean position, the acceleration of the body is proportional to the displacement and with reference to the mean position it is always directed away.

4.3 ANALOGY OF SHM WITH CIRCULAR MOTION

SHM is a special case of circular motion. To understand this; let us consider a particle moving along circle of radius a with centre at O. Let at $t = 0$ it is at X and it goes to P at any instant t making an angle θ with OX [Fig. 4.2].

Here $\theta = \omega t$, where ω is the uniform angular velocity of the particle. Now, if a perpendicular PM is drawn from P to OX then it will give the displacement y of the particle at that instant t. This corresponds to a point P' on the displacement-time curve on the right hand side of Fig. 4.2.

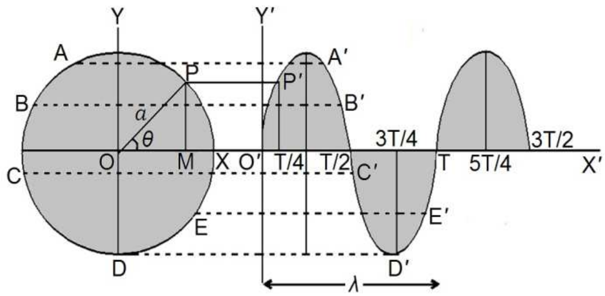

Fig. 4.2: SHM along with its reference circle

Similarly, the displacement of a particle, when it is at A, B, C, D and E in its circular motion, may be represented respectively by A', B', C', D' and E' on the displacement-time curve. Thus after completing a cycle in its circular path the graph on the right side indicates that the motion may be looked upon as simple harmonic. The circle is called the reference circle. From Fig. 4.2 we have,

$$\sin\theta = \frac{PM}{OP} = \frac{y}{a}$$

or,
$$y = a\sin\theta = a\sin\omega t \tag{4.4}$$

If the particle is not at O initially, but at some other point then the equation for the displacement at any instant t is given by,

$$y = a\sin(\omega t + \varphi) \quad \text{[where } \varphi \text{ is the initial phase]} \tag{4.5}$$

Rate of change of displacement of the particle is called its velocity and is given by,

$$v = \frac{dy}{dt} = a\omega \cos(\omega t + \varphi)$$

$$= a\omega\sqrt{1 - \frac{y^2}{a^2}} = \omega\sqrt{a^2 - y^2} \qquad (4.6)$$

So, at the mean position of the motion y is zero and thus the velocity will be maximum.

Now, the rate of change of velocity of the particle is called its acceleration and is given by,

$$f = \frac{dv}{dt} = -a\omega^2 \sin(\omega t + \varphi)$$

$$= -\omega^2 y \qquad (4.7)$$

Thus, at the extreme position of the motion y is maximum and so the acceleration will be maximum and directed towards the mean position.

Example 4.1 Given that the displacement of a particle of mass 0.5 kg under simple harmonic motion is $y = 5\cos\left(\frac{\pi}{3}t - \frac{\pi}{6}\right)$ m. Calculate the

i) amplitude, angular velocity, time period of the particle;

ii) displacement, velocity and acceleration after 1 s;

iii) the kinetic and potential energy when displacement is $x = 0.04$ m; and

iv) the total energy of the particle.

Solution

The equation of a SHM is given by, $y = a\ cos\,(\omega t - \theta)$ (a)

Here $y = 5\cos\left(\frac{\pi}{3}t - \frac{\pi}{6}\right)$ (b)

(i) Comparing Eqs. (a) and (b) we get, the amplitude of SHM is a = 5 m, the angular velocity $\omega = \frac{\pi}{3}$ rad/s and the time period $T = \frac{2\pi}{\omega} = 6$ s.

(ii) At $t = 1$ s the displacement is

$$y = 5\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 5\cos\frac{\pi}{6} = 5\frac{\sqrt{3}}{2}m$$

$$= 4.33 \text{ m.}$$

At any instant, the velocity of the particle is,

$$v = \frac{dy}{dt} = -5\frac{\pi}{3}\sin\left(\frac{\pi}{3}t - \frac{\pi}{6}\right)$$

EXAMPLE 4.1

Thus at $t = 1$ s,

$$v = -5\frac{\pi}{3}\sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right) - 5\frac{\pi}{3}\sin\frac{\pi}{6} = -2.62 \text{ m/s}.$$

Again the acceleration of the particle is given by,

$$f = \frac{dv}{dt} = -5\left(\frac{\pi}{3}\right)^2 \cos\left(\frac{\pi}{3}t - \frac{\pi}{6}\right)$$

Thus at $t = 1$ s,

$$f = \frac{dv}{dt} = -5\left(\frac{\pi}{3}\right)^2 \cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right) - 5\left(\frac{\pi}{3}\right)^2 \cos\frac{\pi}{6} = -4.74 \text{ m/s}^2.$$

(iii) Now the kinetic energy of the particle is given by,

$$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2\left(a^2 - x^2\right)$$

At $x = 0.04$ m, $\text{KE} = \frac{1}{2}\times 0.5\times\left(\frac{\pi}{3}\right)^2\left(5^2 - 0.04^2\right) = 6.845$ J.

and the potential energy of the particle is

$$\text{PE} = \frac{1}{2}m\omega^2 x^2$$

At $x = 0.04$ m, $\text{PE} = \frac{1}{2}\times 0.5\times\left(\frac{\pi}{3}\right)^2 \times(0.04)^2 = 4.38\times 10^{-4}$ J.

(iv) The total energy of the particle is

$$\frac{1}{2}m\omega^2 a^2 = \frac{1}{2}\times 0.5\times\left(\frac{\pi}{3}\right)^2\times(5)^2 = 6.846 \text{ J}.$$

EXAMPLE 4.1

4.4 DIFFERENTIAL EQUATION OF SHM

For a particle undergoing oscillatory motion, if the acceleration is directed to a fixed position along its path and is always proportional to the displacement from that position then the particle motion will be *simple harmonic*. For a particle of mass m executing SHM; if x be the displacement of the particle from the mean position then the force acting on the particle is given by,

$$m\frac{d^2x}{dt^2} \propto -x$$

or,

$$m\frac{d^2x}{dt^2} = -\mu x$$

where μ is the restoring force per unit displacement.

or, $$\frac{d^2x}{dt^2}+\frac{\mu}{m}x=0$$

or, $$\frac{d^2x}{dt^2}+\omega^2 x=0 \tag{4.8}$$

where, $$\omega^2=\frac{\mu}{m}$$

or, $$\mu=m\omega^2$$

Eq. (4.8) is the basic differential equation of a particle executing SHM.

Solution of the equation: Multiplying both sides of Eq. (4.8) by $2\frac{dx}{dt}$ we get,

$$2\frac{dx}{dt}\frac{d^2x}{dt^2}+2\frac{dx}{dt}\omega^2 x=0$$

or, $$\frac{d}{dt}\left\{\left(\frac{dx}{dt}\right)^2+\omega^2x^2\right\}=0$$

or, $$\left(\frac{dx}{dt}\right)^2+\omega^2x^2=K$$

where K is the integration constant.

SCAN ME

for a tutorial on SHM

Now $x=a$, $\frac{dx}{dt}=0$

$\therefore$ $$K=\omega^2a^2$$

So, $$\left(\frac{dx}{dt}\right)^2+\omega^2x^2=\omega^2a^2$$

or, $$\left(\frac{dx}{dt}\right)^2=\omega^2(a^2-x^2)$$

or, $$\frac{dx}{dt}=\omega\sqrt{a^2-x^2}$$

or, $$\frac{dx}{\sqrt{a^2-x^2}}=\omega dt$$

Integrating both sides we get,

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\int\omega dt$$

or, $$\sin^{-1}\frac{x}{a} = \omega t + \varphi$$

or, $$\frac{x}{a} = \sin(\omega t + \varphi)$$

or, $$x = a\sin(\omega t + \varphi) \tag{4.9}$$

which is the required solution.

4.5 HARMONIC OSCILLATOR

It is a vibrating system with its vibration in a simple harmonic manner consisting a mass m attached to a spring of spring constant k as shown in Fig. 4.3. Let the mass m is displaced to point P by a distance x and then released. The restoring force developed will act in the negative direction of x and is given by $$F \propto -x$$

or, $$F = -kx$$

where k is the force constant.

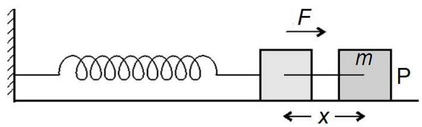

Fig. 4.3: Harmonic Oscillator

or, $$m\frac{d^2x}{dt^2} = -kx \tag{4.10}$$

or, $$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{4.11}$$

So, the motion of the body is simple harmonic. Comparing Eq. (4.11) with the general differential Eq. (4.8) of SHM we get,

$$\omega^2 = \frac{k}{m}$$

or, $$\omega = \sqrt{\frac{k}{m}}$$

$\therefore$ $$T = 2\pi\sqrt{\frac{k}{m}} \tag{4.12}$$

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for more on harmonic oscillator

This gives the time period of the harmonic oscillator.

4.6 DAMPED VIBRATION

In practice it is found that the amplitude of vibration of any vibrating body when left to itself, gradually diminishes and ultimately the amplitude becomes zero. Since it is resisted by different external or internal forces or both, the medium withdraws energy from the vibrating body and the resisting force becomes proportional to the velocity of the particle. Such vibration of particle resisted by damping forces is called *damped vibration*. This type of vibration is experienced by a body in presence of dissipative forces or frictional forces.

4.6.1 Differential Equation

Let a particle of mass m has the displacement x executing damped vibration from its initial position of rest at any instant t. The particle experiences forces which may be written as:

(i) the force of inertia on the body by virtue of its motion and is given by $m\dfrac{d^2x}{dt^2}$,

(ii) force of restitution tending to bring the particle to its initial position of rest and is given by μx; μ being the restoring force per unit displacement, and

(iii) resisting or dissipative force due the damping and is proportional to the velocity of the particle given by $b\dfrac{dx}{dt}$.

Thus at equilibrium the force equation of the particle will be,

$$m\frac{d^2x}{dt^2} = -\mu x - b\frac{dx}{dt}$$

or, $$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{\mu}{m}x = 0$$

or, $$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0 \tag{4.13}$$

where, $$2k = \frac{b}{m}$$

and $$\omega^2 = \frac{\mu}{m}$$

Eq. (4.13) is the general differential equation of motion of a particle executing damped vibration.

Let the general solution of Eq. (4.13) to be,

$$x = ae^{\alpha t}$$

Differentiating twice with respect to t, we get the following two expressions:

$$\frac{dx}{dt} = a\alpha e^{\alpha t} \text{ and } \frac{d^2x}{dt^2} = a\alpha^2 e^{\alpha t}$$

Putting in Eq. (4.13) we have,

$$a\alpha^2 e^{\alpha t} + 2ka\alpha e^{\alpha t} + a\omega^2 e^{\alpha t} = 0$$

or, $$ae^{\alpha t}\left(\alpha^2 + 2k\alpha + \omega^2\right) = 0$$

or, $$\alpha^2 + 2k\alpha + \omega^2 = 0 \qquad [\text{ as } ae^{\alpha t} \neq 0]$$

i.e., $$\alpha = \frac{-2k \pm \sqrt{4k^2 - 4\omega^2}}{2} = -k \pm \sqrt{k^2 - \omega^2}$$

Thus, the general solution of Eq. (4.13) may be written as,

$$x = A_1 e^{\left(-k+\sqrt{k^2-\omega^2}\right)t} + A_2 e^{\left(-k-\sqrt{k^2-\omega^2}\right)t} = e^{-kt}\left[A_1 e^{\sqrt{k^2-\omega^2}\,t} + A_2 e^{-\sqrt{k^2-\omega^2}\,t}\right] \tag{4.14}$$

where A_1 and A_2 are two constants to be evaluated. Depending on the relative values of k and ω there may be three cases –

Case 1 - Under-damped motion with light damping: When $k < \omega$ *i.e.*, the damping force is small then $\sqrt{k^2 - \omega^2}$ is imaginary and is given by,

$$\sqrt{k^2 - \omega^2} = i\sqrt{\omega^2 - k^2}$$

Hence Eq. (4.14) reduces to –

$$x = e^{-kt}\left[A_1 e^{i\sqrt{\omega^2-k^2}t} + A_2 e^{-i\sqrt{\omega^2-k^2}t}\right]$$

or,
$$x = e^{-kt}\left[A_1\left(\cos\sqrt{\omega^2-k^2}t + i\sin\sqrt{\omega^2-k^2}t\right) + A_2\left(\cos\sqrt{\omega^2-k^2}t - i\sin\sqrt{\omega^2-k^2}t\right)\right]$$

or,
$$x = e^{-kt}\left[(A_1 + A_2)\cos\sqrt{\omega^2-k^2}t + i(A_1 - A_2)\sin\sqrt{\omega^2-k^2}t\right]$$

Let, us assume $(A_1 + A_2) = P\sin\theta$

and $i(A_1 - A_2) = P\cos\theta$

$\therefore$
$$x = e^{-kt}\left[P\cos\sqrt{\omega^2-k^2}t\sin\theta + P\sin\sqrt{\omega^2-k^2}t\cos\theta\right]$$

$$= Pe^{-kt}\sin\left(\sqrt{\omega^2-k^2}t + \theta\right) \tag{4.15}$$

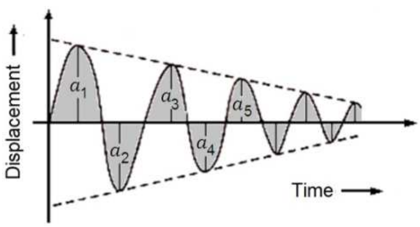

Fig. 4.4: Graphical illustration of damped oscillatory motion

This represents the *damped oscillatory* motion.

The graphical representation is shown in Fig. 4.4. The amplitude of the motion is Pe^{-kt} which decreases exponentially at a rate determined by the decay constant k and the time period of the motion is,

$$T = \frac{2\pi}{\sqrt{\omega^2 - k^2}} \tag{4.16}$$

From Eq. (4.15) it is clear that the presence of the damping force has the following two effects—

(a) The amplitude is no longer constant but dies away according to the factor e^{-kt}.

(b) The frequency reduces slightly below the free natural frequency and is given by $\dfrac{\sqrt{\omega^2 - 2k^2}}{2\pi}$, instead of $\dfrac{\omega}{2\pi}$.

e.g., Motion of a pendulum in air, electric oscillations of LCR circuits *i.e.*, discharge of a charged capacitor through an inductive coil of low inductance, oscillations of coils in ballistic galvanometer *etc.*

Case 2 - Over-damped motion with heavy damping (deadbeat motion):

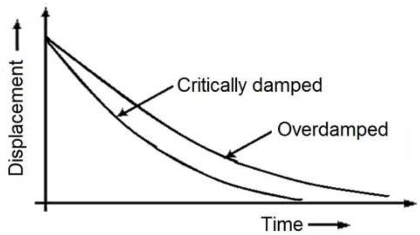

Fig. 4.5: Illustration of critically damped and overdamped motion

When $k > \omega$ *i.e.*, the damping force is large then the motion will be non-oscillatory. In this case $\sqrt{k^2-\omega^2} < k$ and so from Eq. (4.15) we get the displacement x of the body gradually decreases exponentially with time and it comes to its equilibrium position at $t \to \infty$ without performing any oscillation. This type of motion is called *over-damped* or *aperiodic* or *dead-beat motion*. The motion is illustrated in Fig. 4.5.

e.g., The motion of pendulum in a viscous liquid or discharging of capacitor through a resistor.

Case 3 - Critical damping: When $k = \omega$ then we get from Eq. (4.15),

$$x = e^{-kt}\left[A_1 e^{\sqrt{k^2-\omega^2}\,t} + A_2 e^{-\sqrt{k^2-\omega^2}\,t}\right] = e^{-kt}\left[A_1\left(1+\sqrt{k^2-\omega^2}\,t\right)+A_2\left(1-\sqrt{k^2-\omega^2}\,t\right)\right]$$

$$= e^{-\omega t}\left[\left(A_1 + A_2\right) + \left(A_1 - A_2\right)\sqrt{k^2-\omega^2}\,t\right]$$

$$\therefore \qquad x = \left(A + Bt\right)e^{-\omega t} \qquad (4.17)$$

where $A = A_1 + A_2$ and $B = \left(A_1 - A_2\right)\sqrt{k^2-\omega^2}$

Motion of this type is termed as *critically damped motion* which is also illustrated in Fig. 4.5. In this case the damped oscillatory motion suddenly changes into a dead beat motion but the particle tends to move to equilibrium much more rapidly than the overdamped motion. It is also a non-oscillatory motion.

Example 4.2 A particle of mass 0.02 kg is acted on by a restoring force per unit displacement 10×10^{-3} N/m and frictional of force per unit velocity of 2×10^{-3} Nm^{-2}-s. Find

i) whether the motion is oscillatory or not,

ii) for what value of resistive force the damping will be critical and

iii) its time period, if the particle is displaced through 2 cm.

Solution

Here, $\mu = \dfrac{\text{Resisting force}}{\text{Displacement}} = 10\times10^{-3}$ N/m,

$b = \dfrac{\text{Frictional force}}{\text{Velocity}} = 2\times10^{-3}$ Nm^{-1}-s,

and $m = 0.2$ kg.

(i) $k = \dfrac{b}{2m} = \dfrac{2\times10^{-3}}{2\times0.2} = 0.005\ \text{s}^{-1}$.

EXAMPLE 4.2

EXAMPLE 4.2

Thus, $\omega = \sqrt{\dfrac{\mu}{m}} = \sqrt{\dfrac{10\times10^{3}}{0.2}} = 0.22\ \mathrm{s}^{-1}.$

As, $k < \omega$ the motion is oscillatory.

(ii) For the motion to be critically damped, $\sqrt{\dfrac{\mu}{m}} = \dfrac{b}{2m}$

or, $b = \sqrt{4\mu m}$

or, $b = \sqrt{4\times0.2\times10\times10^{-3}} = 8.94\times10^{2}\ \mathrm{Nm^{-1}\text{-}s}.$

(iii) Time period is given by

$$T = \frac{2\pi}{\sqrt{(0.22)^{2} - (0.005)^{2}}} = 28.57\ \mathrm{s}.$$

EXAMPLE 4.3

Example 4.3 A body of mass 10 g is acted upon by a restoring force per unit displacement of 10 dyne-cm^{-1} and resisting force per unit velocity of 2 dyne-cm^{-1}s. Obtain

i) whether the motion is periodic or oscillatory,

ii) the resisting force that makes the movement critically damped and

iii) the mass so that for the applied force makes the motion critically damped.

Solution

Here, $\mu = \dfrac{\text{Resisting force}}{\text{Displacement}} = 10$ dyne/cm,

$b = \dfrac{\text{Frictional force}}{\text{Velocity}} = 2$ dyne-cm^{-1}s and

$m = 10$ g.

(i) $k = \dfrac{b}{2m} = \dfrac{2}{20} = 0.1\ \mathrm{s}^{-1}.$

$\omega = \sqrt{\dfrac{\mu}{m}} = \sqrt{\dfrac{10}{10}} = 1\ \mathrm{s}^{-1}.$

Since, $\omega > k$ therefore the motion is oscillatory.

(ii) For critically damped motion $\sqrt{\dfrac{\mu}{m}} = \dfrac{b}{2m}$

or, $b = \sqrt{4\mu m}$

or, $b = \sqrt{4 \times 10 \times 10} = 20$ dyne-cm^{-1}s.

(iii) Using $b = \sqrt{4\mu m}$

we get, $m = \dfrac{b^2}{4\mu} = \dfrac{20^2}{4 \times 10} = 10$ gm,

which is the required mass for which the given force will make the motion critically damped.

EXAMPLE 4.3

Example 4.4 The equation for displacement of a point of a damped harmonic oscillator is,

$x = 5e^{-0.25t} \sin\left(\dfrac{\pi}{2}\right) t\,\text{m}$. Find the velocity of the oscillating point at $t = \dfrac{T}{4}$, where T is the time period of oscillator.

Solution

Equation for displacement of the damped harmonic oscillator is,

$$x = 5e^{-0.25t} \sin\left(\frac{\pi}{2}\right) t\,\text{m}. \qquad \text{(a)}$$

Comparing the above equation with, $x = Pe^{-kt} \sin \omega t$, we get $\omega = \pi/2$.

Thus, the period is $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\pi/2} = 4$ s.

Now differentiating Eq. (a) with respect to t, we get,

$$\frac{dx}{dt} = 5 \times (-0.25)\, e^{-0.25t} \sin\left(\frac{\pi}{2}\right) + 5\frac{\pi}{2} e^{-0.25t} \cos\left(\frac{\pi}{2}\right)$$

At, $t = \dfrac{T}{4} = \dfrac{4}{4} = 1$ s,

velocity, $v = -1.25e^{-0.25t} \sin\dfrac{\pi}{2} + 5\dfrac{\pi}{2} e^{-0.25t} \cos\dfrac{\pi}{2} = -0.974$ m/s.

So, the velocity is in the opposite direction.

EXAMPLE 4.4

4.7 FORCED VIBRATION

The tendency of an object to apply force to another nearby object into vibrational motion is called *forced vibration*. In forced vibration there must be an external force acted on the system to execute the vibration. This external force may be harmonic, non harmonic, periodic or non-periodic. For instance if some objects are set into vibration at their natural frequency by somehow disturbing the object, the entire system begins to vibrate and forces system surrounding into vibrational motion.

e.g., In musical instruments like guitar string, guitar and enclosed air are forced to vibrate.

4.7.1 Differential Equation

Let an external periodic force acts on a mass m and is represented by $F \cos pt$, where F is the amplitude and p is the angular frequency of the applied force. Let x be the displacement of a particle executing forced vibration from its initial position of rest at any instant t. The forces acting on the particle are,

(i) Force of inertia $m\dfrac{d^2x}{dt^2}$ due to its motion.

(ii) Force of restitution tending to bring the particle to its initial position of rest and is given by μx; μ being the restoring force per unit displacement.

(iii) Damping force which is directly proportional to the velocity of the particle and is given by $b\dfrac{dx}{dt}$, b is the resisting force per unit velocity.

(iv) The external periodic force $F \cos pt$.

Hence, we can write the equation of motion as,

$$m\frac{d^2x}{dt^2} = -\mu x - b\frac{dx}{dt} + F\cos pt$$

or,
$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{\mu}{m}x = \frac{F}{m}\cos pt$$

or,
$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = f\cos pt \tag{4.18}$$

where,
$$2k = \frac{b}{m},\ \omega^2 = \frac{\mu}{m} \text{ and } f = \frac{F}{m}$$

Eq. (4.18) represents the differential equation of motion of a particle of mass m undergoing forced vibration under the action of a periodic force $F \cos pt$.

4.7.2 Solution of the Differential Equation

In case of forced vibration initially the system will try to vibrate with its natural period but ultimately it will vibrate with the period and frequency of the external periodic force and hence the solution of Eq. (4.18) may be written as,

$$x = A\cos(pt - \alpha) \tag{4.19}$$

Differentiating both sides of Eq. (4.19) twice with respect to t we get the following two expressions:

$$\frac{dx}{dt} = -Ap\sin(pt - \alpha)$$

and $$\frac{d^2x}{dt^2} = -Ap^2 \cos(pt - \alpha)$$

Using these in Eq. (4.18) we get,

$$-Ap^2 \cos(pt-\alpha) - 2kAp\sin(pt-\alpha) + \omega^2 A\cos(pt-\alpha)$$
$$= f\cos pt = f\cos(pt - \alpha + \alpha)$$
$$= f\cos(pt-\alpha)\cos\alpha - f\sin(pt-\alpha)\sin\alpha$$

or, $$A\left(\omega^2 - p^2\right) f\cos(pt-\alpha) - 2kAp\sin(pt-\alpha)$$
$$= f\cos(pt-\alpha)\cos\alpha - f\sin(pt-\alpha)\sin\alpha$$

As this is true for all values of t the coefficients of $\cos(pt-\alpha)$ and $\sin(pt-\alpha)$ on both sides must be separately equal. Hence,

$$A\left(\omega^2 - p^2\right) = f\cos\alpha \tag{4.20}$$

and $$2kAp = f\sin\alpha \tag{4.21}$$

Squaring and adding Eq. (4.20) and Eq. (4.21) we get,

$$f^2 = A^2\left(\omega^2 - p^2\right)^2 + 4A^2k^2p^2$$

or, $$A^2 = \frac{f^2}{\left(\omega^2 - p^2\right)^2 + 4k^2p^2}$$

or, $$A = \frac{f}{\sqrt{\left(\omega^2 - p^2\right)^2 + 4k^2p^2}} \tag{4.22}$$

Again dividing Eq. (4.21) by Eq. (4.20) we get,

$$\tan\alpha = \frac{2kp}{\left(\omega^2 - p^2\right)}$$

or, $$\alpha = \tan^{-1}\frac{2kp}{\left(\omega^2 - p^2\right)} \tag{4.23}$$

Thus using Eqs. (4.22) and (4.23) solution for Eq. (4.18) may be written as,

$$x = \frac{f}{\sqrt{\left(\omega^2 - p^2\right)^2 + 4k^2p^2}}\cos\left(pt - \tan^{-1}\frac{2kp}{\left(\omega^2 - p^2\right)}\right) \tag{4.24}$$

This gives the particular solution for the motion having the same frequency as that of the driving force, but lagging behind by an angle α. This is true only for *steady state*. But, at the beginning as the force sets the particle into vibration, the particle will vibrate with its natural frequency. The solution for the displacement is complementary solution which can be obtained by solving equation for damped oscillation,

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0$$

Now, we have the solution for damped oscillation as,

$$x = Pe^{-kt}\sin(\sqrt{\omega^2 - k^2}\,t + \theta)$$

Hence the general solution is given by,

$$x = \frac{f}{\sqrt{\left(\omega^2 - p^2\right)^2 + 4k^2p^2}}\cos\left(pt - \tan^{-1}\frac{2kp}{\left(\omega^2 - p^2\right)}\right)$$

$$+Pe^{-kt}\sin(\sqrt{\omega^2 - k^2}\,t + \theta) \qquad (4.25)$$

4.8 DISCUSSION FOR STEADY STATES

The displacement amplitude and velocity amplitude of a body undergoing forced vibration depends on the frequency of the applied external periodic force. When the displacement amplitude of a particle undergoing forced vibration is maximum for a particular frequency of the applied periodic force, the phenomena is called *amplitude resonance*. The frequency is termed as the resonance frequency.

4.8.1 Amplitude Resonance

In steady state the amplitude of vibration is given from Eq. (4.22) as,

$$A = \frac{f}{\sqrt{\left(\omega^2 - p^2\right)^2 + 4k^2p^2}}$$

When the amplitude becomes maximum, the amplitude resonance takes place. But A will be greatest when the denominator $\sqrt{\left(\omega^2 - p^2\right)^2 + 4k^2p^2}$ will be minimum, because f is fixed. In this case, ω and k are constants and the only variable is p.

So, if the denominator is to be minimum, its differential coefficient with respect to p must be zero.

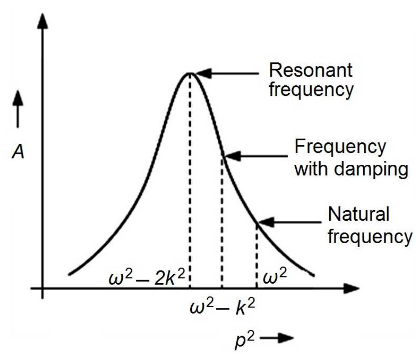

Fig. 4.6: Occurrence of amplitude resonance at resonant frequency

Hence, $$\frac{d}{dp}\left[\left(\omega^2 - p^2\right)^2 + 4k^2p^2\right] = 0$$

or, $$-4p\left(\omega^2 - p^2\right) + 8k^2p = 0$$

or, $$p=\sqrt{\omega^2-2k^2} \tag{4.26}$$

Thus, when the frequency of the driving force is $\frac{1}{2\pi}\sqrt{\omega^2-2k^2}$, then amplitude resonance takes place and the maximum amplitude is given by,

$$A_{\max}=\frac{f}{2k\sqrt{\omega^2-k^2}} \tag{4.27}$$

Hence, the amplitude is greater when the damping factor k is lower and it is illustrated in Fig. 4.6.

4.8.2 Velocity and Energy Resonances

When the velocity amplitude of a body undergoing forced vibration attains a maximum value for a certain frequency of the applied periodic force, the phenomena is known as *velocity resonance*. The motion of a particle in the steady state under the action of a periodic force is obtained from Eq. (4.19) as, $$x=A\cos(pt-\alpha)$$

Hence, $$\frac{dx}{dt}=-Ap\sin(pt-\alpha)$$

Thus the KE of the forced system at any instant is given by,

$$T=\frac{1}{2}m\left(\frac{dx}{dt}\right)^2=\frac{1}{2}mA^2p^2\sin^2(pt-\alpha)$$

$$=\frac{1}{2}\frac{mf^2p^2\sin^2(pt-\alpha)}{\left(\omega^2-p^2\right)^2+4k^2p^2} \tag{4.28}$$

Now, for a steady motion, the total energy of the system at any instant is equal to the maximum KE. Hence the total energy of the system is,

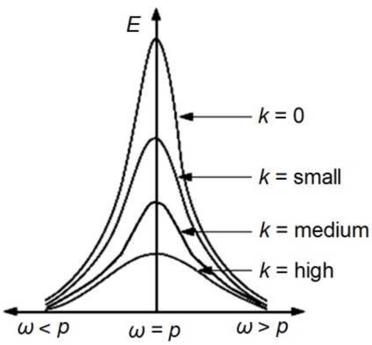

Fig. 4.7: Variation of energy with frequency of the external force

$$E=T_{\max}=\frac{1}{2}\frac{mf^2p^2}{\left(\omega^2-p^2\right)^2+4k^2p^2}$$

or, $$E=\frac{1}{2}\frac{mf^2}{\omega^2\left(\frac{\omega}{p}-\frac{p}{\omega}\right)^2+4k^2}$$

$$=\frac{1}{2}\frac{mf^2}{\omega^2\Delta^2+4k^2} \tag{4.29}$$

where, $$\Delta=\frac{\omega}{p}-\frac{p}{\omega} \tag{4.30}$$

Energy resonance occurs when the energy of the vibrating system is maximum.

Again for the vibrating system total energy becomes same as the maximum KE, which is the case when velocity is maximum. So, this type of resonance is known as velocity resonance. With frequency the variation of energy of the external periodic force is illustrated in Fig. 4.7.

When $\omega = p, \Delta = 0$ the energy becomes highest for any given value of k. Thus when frequency of the forcing system coincides with that of the natural angular frequency f of the forced system, the energy of the forced system is maximum. This phenomena is called energy resonance and the energy at resonance is obtained from Eq. (4.29) as,

$$E_{\max} = \frac{mf^2}{8k^2} \tag{4.31}$$

From Eq. (4.31) it is clear that smaller the value of k, greater is the value of E.

4.8.3 Sharpness of Resonance

At resonant frequency the response of the driving body *i.e.*, the energy taken by it from the drived force is maximum. But this will diminish if the frequency differs from the resonant frequency. The rapidity with which the response falls is a measure of *sharpness of resonance*. The sharpness of resonance is quantitatively defined as the reciprocal of Δ for which the energy is reduced to half the resonance value.

Let for $\Delta = \Delta_1$, $E = \frac{1}{2}E_{\max} = \frac{1}{2}\frac{mf^2}{8k^2}$

Thus, using Eq. (4.29) and putting $\Delta = \Delta_1$, we have,

$$\frac{1}{2}\frac{mf^2}{\Delta_1^2\omega^2 + 4k^2} = \frac{1}{2}\frac{mf^2}{8k^2}$$

or, $$\Delta_1^2\omega^2 + 4k^2 = 8k^2$$

or, $$\Delta_1 = \pm\frac{2k}{\omega} \tag{4.32}$$

Hence sharpness of resonance

$$S = \frac{1}{\Delta_1} = \pm\frac{\omega}{2k} \tag{4.33}$$

From Eq. (4.33) it is clear that sharpness of resonance increases as the damping factor decreases and for a particular value of k it is greater as higher the value of natural frequency ω. The sharpness of resonance can be more rapidly represented by plotting $\frac{E}{E_{\max}}$ against ω.

When, $\omega = p$, $\Delta = 0$ we have, $\frac{E}{E_{\max}} = 1$

It is clear from the curves that smaller the value of damping factor k, the sharper is the resonance.

Example 4.5 A body of mass 10 g is acted upon by a restoring force per unit displacement 10^7 dyn-cm^{-1}, a frictional force per unit velocity 4×10^3 dyne-cm^{-1}s and a driving force of $10^5 \cos pt$ dyne. Find the value of maximum amplitude.

Solution

The amplitude of a particle executing forced vibration is given by,

$$A = \frac{f}{\sqrt{\left(\omega^2 - p^2\right)^2 + (2kp)^2}} = \frac{F/m}{\sqrt{\left(\omega^2 - p^2\right)^2 + (2kp)^2}}$$

When, $p = \omega$,

$$A = A_{\max} = \frac{F/m}{\left(\sqrt{2kp}\right)^2} = \frac{F}{2kmp}$$

Here, $$k = \frac{b}{2m} = \frac{4\times10^3}{2\times10} = 2\times10^2\ \text{s}^{-1}.$$

$$p = \omega = \sqrt{\frac{\mu}{m}} = \sqrt{\frac{10^7}{10}} = 10^3\ \text{s}^{-1}.$$

Thus, $$A_{\max} = \frac{10^5/10}{2\times2\times10^3\times10^3} = \frac{1}{40} = 0.025\ \text{cm}.$$

EXAMPLE 4.5

UNIT SUMMARY

- ***Simple harmonic motion (SHM)***

  $F = -\mu x$

  Displacement, velocity and acceleration of SHM

  $y = a\sin(\omega t + \varphi)$

  $v = \dfrac{dy}{dt} = \omega\sqrt{a^2 - y^2}$

  $f = \dfrac{dv}{dt} = -\omega^2 y$

- ***Differential equation of SHM and its solution***

  $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$

  $x = a\sin(\omega t + \varphi)$

- ***Harmonic oscillator: Motion of a horizontal spring***

  $\omega = \sqrt{\dfrac{k}{m}}$ $\qquad$ $T = 2\pi\sqrt{\dfrac{k}{m}}$

- ***Differential equation of damped vibration***

  $\dfrac{d^2x}{dt^2} + 2k\dfrac{dx}{dt} + \omega^2 x = 0$ $\qquad$ where $2k = \dfrac{b}{m}$; $\omega^2 = \dfrac{\mu}{m}$

- ***Differential equation of forced vibration***

  $\dfrac{d^2x}{dt^2} + 2k\dfrac{dx}{dt} + \omega^2 x = f\cos pt$ $\quad$ where $2k = \dfrac{b}{m}$; $\omega^2 = \dfrac{\mu}{m}$; $f = \dfrac{F}{m}$

- ***Solution of the differential equation of damped vibration***

  $$x = \frac{f}{\sqrt{\left(\omega^2 - p^2\right)^2 + 4k^2p^2}}\cos\left(pt - \tan^{-1}\frac{2kp}{\left(\omega^2 - p^2\right)}\right)$$

  $$+Pe^{-kt}\sin(\sqrt{\omega^2 - k^2}\,t + \theta)$$

- ***Resonances***

  Amplitude resonance

  $A_{\max} = \dfrac{f}{2k\sqrt{\omega^2 - k^2}}$

  Energy resonance

  $E = \dfrac{1}{2}\dfrac{mf^2}{\omega^2\Delta^2 + 4k^2}$ $\qquad$ $E_{\max} = \dfrac{mf^2}{8k^2}$

Sharpness of resonance

$$S = \frac{1}{\Delta_1} = \pm \frac{\omega}{2k}$$

EXERCISES

Multiple Choice Questions

4.1 If the restoring force constant of a body is 98 Nm^{-1}, then the restoring force of the body for a displacement of 10 cm is

(a) 98 N (b) 9.8 N (c) 0.98 N (d) none of these

4.2 The acceleration of a particle executing SHM is

(a) inversely proportional to the displacement from the mean position

(b) directly proportional to the displacement from the mean position

(c) constant (d) none of these

4.3 If acceleration of a particle executing SHM is maximum, displacement will be

(a) maximum (b) minimum (c) zero (d) none of these

4.4 Simple Harmonic Motion (SHM) is a

(a) periodic motion (b) circular motion (c) both (a) and (b) (d) none of these

4.5 Natural angular frequency of a body executing SHM is

(a) ω (b) $\omega/2\pi$ (c) $2\pi/\omega$ (d) none of these

4.6 Time period of a horizontal spring (force constant k) can be expressed as,

(a) $T = 2\pi\sqrt{k/m}$ (b) $T = 2\pi\sqrt{m/k}$ (c) $T = \pi\sqrt{k/m}$ (d) none of these

4.7 Time-period (T) of a particle executing SHM with displacement x and acceleration f is

(a) $2\pi\sqrt{\frac{f}{x}}$ (b) $2\pi\sqrt{xf}$ (c) $2\pi\sqrt{\frac{x}{f}}$ (d) none of these

4.8 The effect of the damping force on a harmonic oscillator is to

(a) reduce angular frequency and amplitude of vibration

(b) increase amplitude and angular frequency of vibration

(c) reduce angular frequency and increase amplitude of vibration

(d) increase angular frequency and reduce amplitude of vibration

4.9 For oscillatory motion of damped oscillator (damping constant γ and natural frequency ω_0)

(a) $\gamma > 2\omega_0$ (b) $\gamma = 2\omega_0$ (c) $\gamma < 2\omega_0$ (d) $\gamma = \omega_0$

4.10 The time period of a damped vibration is

(a) $2\pi\sqrt{\omega^2 - k^2}$ (b) $2\pi\sqrt{k^2 - 2\omega^2}$ (c) $\dfrac{2\pi}{\sqrt{\omega^2 - k^2}}$ (d) $\dfrac{2\pi}{\sqrt{2k^2 - \omega^2}}$

4.11 For critical damping, the motion of a damped vibration will be non-oscillatory when

(a) $\omega^2 > k^2$ (b) $\omega^2 < k^2$ (c) $\omega = k$ (d) none of these

4.12 Motion of a system in critical damped condition is

(a) damped oscillatory (b) oscillatory (c) harmonic (d) non-oscillatory

4.13 In case of damped vibration the damping force is proportional to its –

(a) velocity (b) displacement (c) acceleration (d) none of these

4.14 In case of critical damping the system is –

(a) non-oscillatory (b) oscillatory (c) vibratory (d) none of these

4.15 Example of weakly damped harmonic oscillator is

(a) dead-beat galvanometer (b) tangent galvanometer

(c) ballistic galvanometer (d) discharge of a capacitor through a resistance

4.16 The phase difference between external force and velocity of the forced oscillator is

(a) $\dfrac{\pi}{2}$ (b) 0 (c) π (d) none of these

4.17 At resonance, the maximum amplitude of the forced oscillator is

(a) $\dfrac{F_0}{m\gamma}$ (b) $\dfrac{mF_0}{\gamma}$ (c) $\dfrac{\gamma}{mF_0}$ (d) $mF_0\gamma$

Answers of Multiple Choice Questions

4.1 (b), 4.2(b), 4.3 (a), 4.4 (c), 4.5 (b), 4.5 (b), 4.7 (c), 4.8 (a), 4.9 (c), 4.10 (c), 4.11 (c), 4.12 (d), 4.13 (a), 4.14 (a), 4.15 (c), 4.16 (b), 4.17 (a)

Short and Long Answer Type Questions

Category I

4.1 Establish the differential equation of a simple harmonic motion and solve the equation.

4.2 Show that for a particle executing SHM, the average kinetic energy is half of the corresponding maximum energy.

4.3 Derive the expression for the total energy of a simple harmonic oscillator and show that it is constant and proportional to the square of the amplitude.

4.4 Find the relation between time period and frequency.

4.5 Distinguish between damped and forced vibration.

4.6 Find the difference between phase and epoch in relation to SHM.

4.7 Establish the differential equation of damped harmonic oscillator explaining each term in the equation.

4.8 Clarify the term critical damping.

4.9 Distinguish between free and damped vibration. Show graphically the variation of amplitude of damped oscillation with time.

4.10 Interpret the early stage and steady state of forced vibration. Discuss the factors characterizing them.

4.11 Clarify the terms resonance and resonant frequency of a forced oscillator. Give an idea of amplitude resonance.

4.12 Distinguish between amplitude resonance and velocity resonance.

4.13 Under the condition of under damping ($b < \omega$), find the expression for displacement as a function of time. Draw the displacement-time graph and make your conclusions.

4.14 (a) Write down the differential equation of forced vibration explaining each term in the equation.

(b) Starting from the steady state solution of forced vibration explain the phenomenon of amplitude and velocity resonance, deriving the value of the driving frequency in each case.

Category II

4.15 Write down the differential equation of a series LCR circuit driven by sinusoidal voltage. Identify the natural frequency of the circuit and also find the relaxation time.

4.16 (a) Show that at velocity resonance, the maximum velocity is inversely proportional to damping constant and also show that the velocity of the oscillator is in phase with the driving force.

(b) Establish how the half power frequencies are related to sharpness of resonance Show that the resonant frequency is the geometric mean between the half power frequencies.

4.17 Explain how the principle of conservation of energy hold good in case of damped vibration.

4.18 (a) Plots the displacement amplitude and velocity amplitude against the frequency of the sinusoidal force driving a mechanical oscillator for the different values of damping.

(b) Find the energy of an oscillator at velocity resonance. Elaborate the term sharpness of resonance.

4.19 (a) Given that two simple harmonic oscillators of different masses oscillate separately under the action of same restoring force at frequencies 3 Hz and 5 Hz. Calculate the ratio of their masses.

(b) Find the amplitude, phase and instantaneous velocity of the vibrational motion represented by

$$x = a\cos\omega t + \frac{a}{2}\cos\left(\omega t + \frac{\pi}{2}\right) + \frac{a}{4}\cos\left(\omega t + \pi\right) + \frac{a}{8}\cos\left(\omega t + \frac{3\pi}{2}\right).$$

4.20 For a harmonic oscillator of mass m, natural frequency ω_0 and driven by a force F_0 sin ωt, the damping is proportional to $2p$ times the velocity of the oscillator.

Show that the displacement is given by

$$x = \frac{F_0 / m}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4p^2\omega^2}} \sin\left(\omega t - \theta\right); \text{ where } \tan\theta = \frac{2\omega p}{\left(\omega_0^2 - \omega^2\right)}$$

Using the above relation, show that at velocity resonance, the velocity is in phase with the driving force.

Numerical Problems

4.1 The displacement of any particle at any instant t is given by $x = 3\cos t + 4\sin t$. Show that the motion is simple harmonic. What is the amplitude of this oscillation? Show that its kinetic energy oscillates with angular frequency 2ω.

4.2 The displacement of a simple oscillator is given by $x = a\sin(\omega t + \theta)$. If the oscillations started at time $t = 0$ from a position x_0 with velocity v_0 show that,

$$\tan\theta = \omega x_0/v_0 \text{ and } a = (x_0^2 + v_0^2/\omega^2)^{1/2}.$$

4.3 The displacement of a particle of mass 0·2 kg executing SHM is indicated by

$$y = 10\sin\left(\frac{\pi}{3}t - \frac{\pi}{12}\right) \text{ m.}$$

Calculate the (i) amplitude, (ii) angular velocity, (iii) time period, (iv) maximum velocity and (v) maximum acceleration.

4.4 A mass of 2 kg hangs from a spring. A 50 gm body hung below the mass stretches the spring 1 cm further. If now the mass is removed then find the period of oscillation of the body.

4.5 In one dimensional motion of a mass of 10 g, it is acted upon by a restoring force per unit displacement = 10 dyn.cm^{-1} and a resisting force per unit velocity = 2 dyn.cm^{-1}s.

(i) Find whether the motion is aperiodic or oscillatory.

(ii) Find the value of resisting force which will make the motion critically damped.

(iii) Find the value mass for which the given forces will make the motion critically damped.

4.6 For the motion of a damped vibratory body, the restoring force per unit displacement is 10 dyne/cm and a resisting force per unit velocity is 2 dyne-cm^{-1}s. Find whether the motion is oscillatory or aperiodic and also find the resisting force to make the motion critically damped.

4.7 The displacement of a damped harmonic oscillator is given by, $x = 2e^{-t}\sin\pi t$. Find the velocity of the oscillating point at $t = T/2$, T being the time period of the motion.

4.8 A particle of mass 2 kg is acted on by a restoring force/unit displacement 10×10^{-3} N/m, and a frictional force/unit velocity 2×10^{-3} N/ms^{-1}. Find whether the motion is oscillatory or non-oscillatory. For what value of frictional (resistive) force, the damping will be critical?

4.9 The displacement of a particle of mass 0.05 kg executing SHM is indicated by

$$y = 5\sin\left(\pi t - \frac{\pi}{3}\right) \text{ m.}$$

Calculate the (i) amplitude, (ii) angular velocity and (iii) time period.

PRACTICALS

1. To investigate the resonance phenomena in mechanical oscillators

Aim

i) To make an experiment we have to demonstrate the natural frequency of resonance in a mechanical system

ii) To make a comparison of the experimental data with the theory

Theory

Mechanical resonance is essentially tendency of any mechanical system for responding at higher amplitude when its oscillating frequency is in accordance with the natural frequency of vibration of the system. The motion of a particle or a system of particles is periodic or oscillatory if it repeats itself in equal time intervals. The paths of an oscillating particle are usually expressed by sine and cosine functions (harmonic functions). The oscillation of the particle is called harmonic motion. For simple harmonic motion, the position of the particle as a function of time can be expressed in the following form:

$$x = A\cos(\omega t + \varphi) \qquad (i)$$

The constant A represents the amplitude of the motion; the constant ω is for the angular frequency while φ is for the phase constant of the motion. Angular frequency and linear frequency are related as,

$$\omega = 2\pi f \qquad (ii)$$

In the equation, the repetition rate of the motion (frequency) is f, whose reciprocal is the time period, T. In vibrating string on a musical instrument, the time period is very small and so cannot be measured with eye only. The oscillating frequency is then obtained using digital frequency meter attached to a function generator.

For the experiment, simple harmonic oscillator has a mass connected to ideal, mass less spring following Hooke's law. The opposite end of the spring is kept at a fixed position. When the mass is at equilibrium position, there will be no motion. When the system is perturbed, displacing the mass to from equilibrium, restoring force of the spring returns the mass to equilibrium. Inertia is responsible to overshoot the equilibrium causing the oscillatory motion. In absence of friction, the motion would continue.

Using Eq. (ii) and Newton's second law we can derive an expression for ω in term of the mass, m and the spring constant k, we can write

$$\omega = \sqrt{\frac{k}{m}} \qquad (iii)$$

or

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (iv)$$

In this experiment, we will look at a phenomenon known as resonance. If the driving force frequency is identical to natural frequency, the highest displacement of the mass from equilibrium will occur. Resonance will be attained under this condition, and this frequency is called the resonant frequency.

Procedure

First Part: *Determination of the spring constant k*

Make the experimental arrangement as shown in Fig. (*i*). Turn on the air source to the maximum and then loadthe mass hanger with different masses. Record each mass as well as the elongation due to that mass, in Table 1. Add the mass of the hanger and then compute the force due to the mass (*i.e.*, *mg*). Record those values in Table 1.

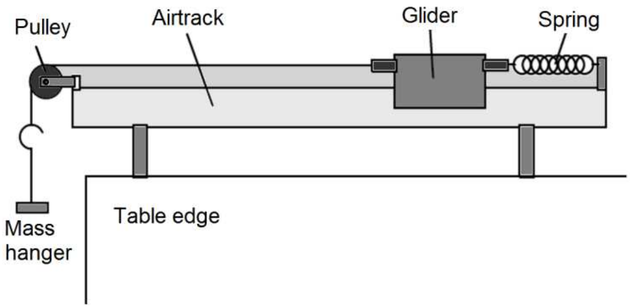

Fig. (i)

In order to find the displacement, note the position of the equilibrium of the spring and also the stretched position. The displacement represents the difference between the two values. Take care not to displace the spring more than 3 or 4 cm from its equilibrium position as otherwise it will deform the spring and will alter its spring constant.

Table 1: Data for displacement and force for different masses

Equilibrium position: ……………..

Mass	Stretched position	Displacement	Force

Spring constant k = ……………….

Drawing of graph: Plot a graph of Elongation (X-axis) *vs.* Force (Y-axis), using the data from table in a graph paper. From the data and graph, determine the spring constant, *k*, of the spring. Theoretically, the curve fitted with this data should be a straight line; *k* is simply the slope of this curve. You can also calculate the slope by fitting a line to the data using Excel.

Second Part: ***Determination of resonance by experiment***

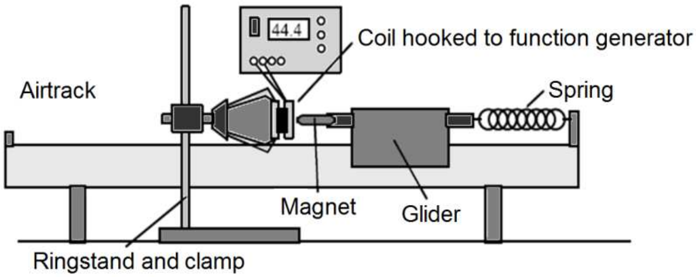

Fig. (ii)

The setup for this part of the experiment is presented in Fig. (*ii*). Driving force represents a sinusoidal electronic function generator with a frequency meter to give the driving frequency. Output of the function generator makes a varying electric current within the coil which in turn creates a varying magnetic field to attract and repel a permanent magnet/glider connected to a spring. Other end of the spring is kept to the air track. We can get the spring constant value (k) and the mass of the magnet/glider directly. Natural frequency of the system can be obtained, in addition to the driving frequency and resonance frequency of the system.

Procedure

1. Turn on the air supply to highest air pressure. Then stabilize the glider and other instruments for turning off the record of the equilibrium position of the glider. Next, turn on the other instruments and before taking any data, use the frequency adjustment control to scan through a range of driving frequencies (1 Hz to 15 Hz). Observe the changes in the glider/magnet movements on the air track. Resonance of this mechanical system occurs at a frequencywhere the highest linear displacement of the glider/magnet is found. Locate this resonance point by tuning the frequency generator until the phenomenon is noted.

2. Getting the knowledge of how resonance happens in the system, start recording the data and put those in Table 2. Vary the driving frequency slowly. Note the amplitude of the motion for each frequency. In the experiment we assume that the excitation signal and the motion of the magnet/glider assembly are symmetrical about the equilibrium.

For each frequency, obtain the glider distance which represents the amplitude. Note the data in Table 2. In order to get a better resonant frequency, take about 10 data sets at frequencies slightly more than the resonant frequency and similarly 10 set at frequencies slightly less than the resonant frequency.

3. To guess better take the initial scanning of the driving frequencies as well as the pick frequencies above and below the guessed value.

4. As the oscillating frequency of glider is very fast, to read the highest displacement by eye a good way is the "index card method". For this you have to slide an index card along the air track close to the glider. Note the position of the card where the glider is about to hit it. The amplitude of the motion is then the difference between the respective position and the equilibrium point. You have to note the glider mass and magnet which is essential for later calculations:

Table 2: Data for frequency vs. amplitude

Mass of glider and magnet: ………………..

Equilibrium position (same as in Table 1): ……………

Frequency	Card Position	Amplitude

Drawing of Graph

Plot the measured amplitudes (Y-axis) as a function of frequency in another graph.

Viva-Voce Questions

1. What are the things might make the resonant frequency alter? Would it alter if the glider given was heavier? Would it alter if there was greater friction between the glider and the air track? Would it alter if the *k* values were not same?

2. What are the things might alter the shape of the plotted curve? Is it mass dependent? Would it alter if the spring constant is not same?

3. Explain the role of resonance that could have played in the oscillations of the bridge for damaging.

4. Is there any frequency for getting the highest amplitude?

5. For a given *k* you measured the first part of the experiment, determine the expected value of resonant frequency.

6. Make a comparison of measured value of the resonant frequency with calculated value. What is the difference in percent between the two measurements?

7. What is resonance?

8. What is the difference between forced vibration and resonance?

2. To investigate the normal modes in coupled oscillations of two bodies

Aim

To find the normal modes of oscillation for two coupled pendulums and to determine the normal mode frequencies

Apparatus

Two compound pendulums, Filament bulb on stand, Screen on stand, Coupling spring, Convergent lens, Stop clock

Theory

Two similar compound pendulums are coupled using a spring. Normal mode oscillations are then excited to find their frequencies. In any coupled system the individual oscillators may have independent natural frequencies. A normal mode motion will be one where all individual oscillators oscillate with identical frequency, known as normal mode frequency. There is a certain phase relationship between the individual motions.

If a system has n degrees of freedom for n coupled oscillators then there are n normal modes of the system. The system disturbance is taken as a superposition of normal mode vibrations. When a single oscillator is excited, the energy is transferred eventually to all the different modes.

In the experiment we have to investigate some features of two compound pendulums (coupled). The system to be considered consists of two similar rigid pendulums, A and B and a linear spring couples the oscillations of the two pendulums. A schematic diagram is presented in Fig. (i).

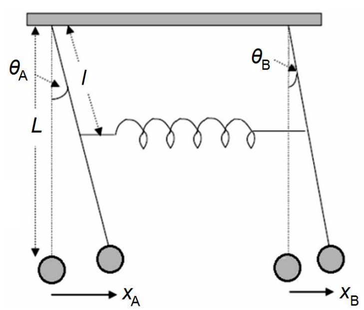

Fig. (i)

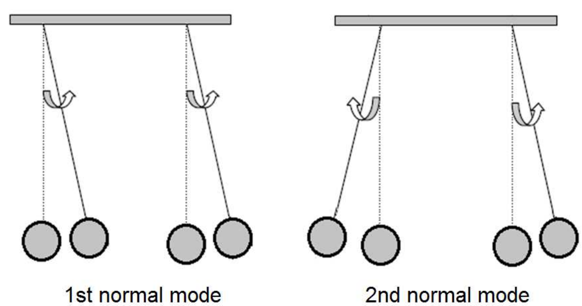

Fig. (ii)

Fig. (ii), on the other hand, illustrates the situation of the first normal mode and the second normal mode. The motion of the pendulum A and pendulum B can be modeled by the coupled differential equations. Let θ_A and θ_B are the angular displacements of the pendulums A and B while I represents the moments of inertia.

Procedure

1. Uncouple the pendulums. Set small oscillations of both pendulums individually. Record the time for 20 oscillations to find the average time period and the natural frequency (ω_0) for free oscillation.

2. By hooking the spring, couple the pendulums at some position to the vertical rods. Be sure that the spring is horizontal and is neither extended nor hanging loose.

3. Switch on the bulb and find the spot at the central position of the screen.

4. By displacing both the pendulums, excite the first normal mode by equal amount in the same direction and then release both the pendulums from rest. The spot on the screen will oscillate in the horizontal direction.

5. Record the time for 20 oscillations for getting the time period T_1 and frequency ω_1 of the first normal mode.

6. Keeping the spring at the same position excite the second normal mode of oscillation displacing both the pendulums in the opposite directions by equal amount and then release them from the rest condition.

7. The spot on the screen now oscillates in the vertical direction. Find the time for 20 oscillations and hence the time period T_2 and frequency ω_2.

8. Repeat the measurements for the spring hooked at 3 more locations on the vertical rod of the pendulums.

9. For some position of the spring, displace one pendulum by small amount, maintaining the other pendulum at its equilibrium and release it from rest. Record the subsequent motion of the pendulums and then qualitatively correlate the motion with the graph [Fig. (*iii*)]. Fig. (*iii*) (a) depicts the plots of (a) $x(t) = \sin(2\pi t)\sin(50\pi t)$ vs. t and Fig. (*iii*) (b) depicts the plot of $x(t) = \cos(2\pi t)\cos(50\pi t)$ vs. t.

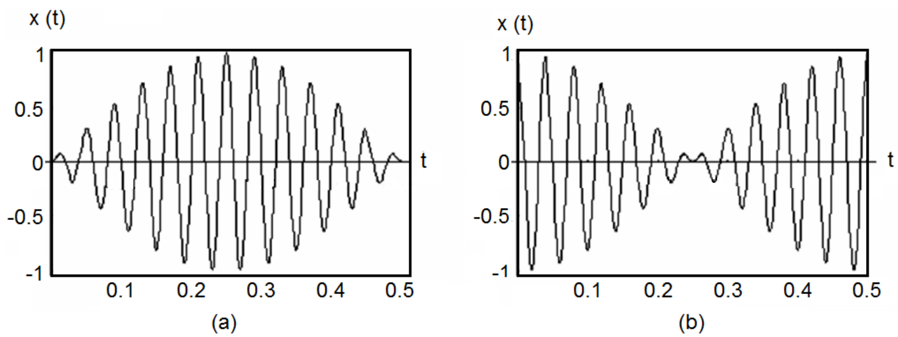

Fig. (iii)

10. Determine the time period T corresponding to individual oscillations of the pendulum A and also the period δT. Infer the time periods T_1 and T_2 of the normal modes from T and δT and compare with previous results.

Viva-Voce Questions

1. Explain the normal mode frequencies for the coupled pendulum considering the motion of the two modes.

2. Discuss why the frequency of the first normal mode is independent of spring position.

3. What do you mean by the normal mode oscillations of a system?

4. What are the number of normal modes will the system posses?

5. What the effect of damping on the motion?
6. Redraw Fig. (*iii*) qualitatively when damping is present.
7. Mention some more examples of coupled oscillations.
8. Discuss the probable sources of random errors at the time of conducting an experiment.
9. Give an interpretation of the movement of the spot on the screen when a single pendulum is displaced.

KNOW MORE

In mechanics simple harmonic motion is periodic motion and it results in an oscillation which without friction or any other dissipation of energy continues indefinitely. Vibration is a mechanical phenomenon where oscillations occur about an equilibrium point.

Activity

Oscillation is essentially the repetitive variation with time about a central value called equilibrium position. The word vibration is precisely used for describing mechanical oscillation. Well known examples of oscillation are swinging pendulum and alternating current. Besides mechanical systems oscillations occur also in dynamic systems and in many areas of science.

e.g., the beating of the human heart, vibration of strings in guitar and other string instruments, periodic firing of nerve cells in the brain, business cycles in economics, predator–prey population cycles in ecology, geothermal geysers in geology, and the periodic swelling of Cepheid variable stars in astronomy.

Interesting facts

All the real-world oscillator systems are thermodynamically irreversible, *i.e.*, there are dissipative processes like friction or electrical resistance. These continually convert some of the energy stored in the oscillator into heat in the environment called damping. Hence oscillations tend to decay with time. An oscillating system may be subjected to some external force. For example, an AC circuit is connected to an outside power source when the oscillation is said to be *driven*.

Analogy

Some of the systems can be excited by transfer of energy from the environment which typically occurs where systems are embedded in fluid flow.

As for example, the phenomenon of flutter in aerodynamics occurs when from its equilibrium an arbitrarily small displacement of an aircraft wing results in an increase in the angle of attack of the wing on the air flow and a consequential increase in lift coefficient, leading to a greater displacement.

At sufficiently large displacements, the stiffness of the wing dominates for providing the restoring force that enables an oscillation.

History

Origin of the science of acoustics is attributed, in general, to the Greek philosopher Pythagoras (6th century BC), whose experiments on the vibrating strings produced pleasing musical intervals that led to a tuning system. In the 6th century AD, the Roman philosopher Boethius documented some ideas relating science to music. He suggested that the human perception of pitch is related to the physical property of frequency.

Timelines

2600 BC: In an exhibition the British Museum exhibits an 11-stringed harp with a gold-decorated, bull-headed sounding box.

3000 BC: Stringed instruments such as harps were depicted on walls of Egyptian tombs. Our present system of music is largely based on ancient Greek civilization.

4000 BC: Music had become highly developed and was much appreciated by Chinese, Hindus, Japanese and the Egyptians. They observed certain rules in connection with the art of music.

1609: *Galileo Galilei* became the first man to point a telescope to the sky. His works on the oscillations of a simple pendulum and the vibration of strings are of fundamental significance in the theory of vibrations.

1714: *Brook Taylor* derived the fundamental frequency of a stretched vibrating string.

1733: *Daniel Bernoulli* derived the fundamental frequency and harmonics of a hanging chain.

1734: *Daniel Bernoulli* solved the ordinary differential equation for the vibrations of an elastic bar.

1739: *Leonhard Euler* solved the ordinary differential equation for a forced harmonic oscillator.

1759: *Leonhard Euler* solved the partial differential equation for the vibration of a rectangular drum.

1764: *Leonhard Euler* examined the partial differential equation for vibration of a circular drum and established one of the Bessel function solutions.

Applications (Real Life / Industrial)

Mathematics of oscillation essentially deals with the quantification of the amount that a sequence or function tends to move between extremes. There are some related notions, like oscillation of a sequence of real numbers, oscillation of a real valued function at a point, and oscillation of a function on an interval.

Typical acoustic applications include acoustic ranging, acoustic location, process control, SONAR, seismology, acoustic emission, vibration analysis, engine testing, ocean acoustic tomography and bio-acoustics.

Case Study (Environmental / Sustainability / Social / Ethical Issues)

Environmental acoustics is associated with noise and vibration caused by railways, road traffic, aircraft, industrial equipment and recreational activities. The main purpose of these studies is to minimize the levels of environmental noise and vibration. Research work also has a focus on the positive use of sound in urban environments, soundscapes and tranquility.

Resonance phenomena is found to occur with different types of vibrations producing acoustic resonance, nuclear magnetic resonance (NMR), electromagnetic resonance, electron spin resonance (ESR), mechanical resonance and resonance of quantum wave functions. Resonant systems are utilized for generating vibrations of a particular frequency or it can pick out some specific frequencies from a complex vibration system having many frequencies. The term *resonance* was originated from the field of acoustics.

In radio engineering and electronics engineering, the approximate symmetric response is popularly named as the *universal resonance curve* which is a concept introduced by Frederick E. Terman in 1932 for simplifying approximate analysis of radio circuits with a range of center frequencies and Q values.

NMR Magnet at Birmingham, UK, with strong 21.2-tesla field, proton resonance is at 900 MHz

Inquisitiveness and Curiosity Topics

Noise and life quality are correlated. The increase of environmental noise, particularly for those living near railways and airports, has created conflict. Getting adequate concentration and quality sleep is very difficult for those who live in areas of high noise exposure. When the body is at rest, a noise stimulus is continually being presented in the environment. The body responds to this noise which can negatively affect sleep. High exposure to environmental noise can play a major role in cardiovascular disease. Noise can raise blood pressure, change heart rate and release stress hormones which can lead to risks for hypertension, arteriosclerosis and even more serious events such as a stroke or myocardial infarction.

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5 Rotational Motion

UNIT SPECIFICS

The following topics are considered in this unit:

- Definition and motion of a rigid body in a plane;
- Rotation in a plane;
- Kinematics in a coordinate system rotating and translating in the plane;
- Angular momentum about a point of a rigid body in planar motion;
- Euler's laws of motion, their independence from Newton's laws, and their necessity in describing rigid body motion;
- Examples of rotational motion.

The practical applications of the topics are discussed for generating further curiosity and creativity as well as improving problem solving capacity.

Besides giving a large number of multiple choice questions as well as questions of short and long answer types marked in two categories following lower and higher order of Bloom's taxonomy, assignments through a number of numerical problems, a list of references and suggested readings are given in the unit so that one can go through them for practice. It is important to note that for getting more information on various topics of interest some QR codes have been provided in different sections which can be scanned for relevant supportive knowledge.

After the related practical, based on the content, there is a "Know More" section. This section has been carefully designed so that the supplementary information provided in this part becomes beneficial for the users of the book. This section mainly highlights the initial activity, examples of some interesting facts, analogy, history of the development of the subject focusing the salient observations and finding, timelines starting from the development of the concerned topics up to the recent time, applications of the subject matter for our day-to-day real life or/and industrial applications on variety of aspects, case study related to environmental, sustainability, social and ethical issues whichever applicable, and finally inquisitiveness and curiosity topics of the unit.

Rationale

This unit on rotational motion will help our students to get a clear idea about some basic terminologies of rigid body. Kinematics in a coordinate system rotating and translating in the plane as well as the angular momentum about a point of a rigid body in planar motion have special importance for all practical purposes. Euler's laws of motion, their independence from Newton's laws and their necessity in describing rigid body motion have wide area of applications which are also critically focussed.

Rotational motion is a motion of an object around a circular path but in a fixed orbit. It is the motion of a body where all of its particles move in a circular motion with a common angular velocity, about a fixed point. As for example, we can mention the rotation of Earth about its axis. In a circular motion, the object just moves in a circle while in rotational motion, the object rotates about an axis. Motion of a Ferris wheel in an amusement park is an example of rotational motion.

PRE-REQUISITES

Mathematics: Preliminary Geometry (Class X)

Physics: Rotational Motion (Class XII)

UNIT OUTCOMES

List of outcomes of this unit is as follows:

U5-O1: Explain the analogy between different linear and angular quantities

U5-O2: Describe kinematic equations for rotational motion

U5-O3: Describe the importance of Euler's equation in rotational motion problems

U5-O4: Calculate moment of inertia of regularly shaped bodies about a fixed axis

U5-O5: Realize the role for rotational inertia to resist rotational motion

U5-O6: Apply the concept of torque and angular momentum to rotational motion problems

Unit-5 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (*1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation*)					
	CO-1	CO-2	CO-3	CO-4	CO-5	CO-6
U5-O1	2	2	1	-	-	-
U5-O2	1	2	1	-	-	-
U5-O3	3	2	1	-	-	-
U5-O4	2	2	1	-	-	-
U5-O5	2	2	1	-	-	-
U5-O6	2	3	1	-	-	-

5.1 INTRODUCTION

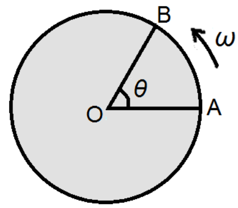

Fig. 5.1: Rotational motion

When a particle moves round a point in a plane in such a way that the distance of the particle about that point remains constant, it is said to perform a circular motion. Here the fixed point is known as the centre of the motion and the distance of the particle from this fixed point is the radius of the circular path. When a body is performing a circular motion or a rotational motion, there is a fixed axis perpendicular to the plane of rotation and is called the axis of rotation.

5.1.1 Angular Displacement

The angular displacement of a particle in a fixed interval of time undergoing circular motion is defined as the angle swept out by the radius vector in that interval. It is given by,

$$\theta = \frac{S}{r} \tag{5.1}$$

where S is the linear displacement of the particle while moving from A to B (Fig. 5.1) and is given by,

$$S = r\,\theta \tag{5.2}$$

5.1.2 Angular Velocity

It represents the rate of variation of the angular displacement of a body. This means it is the angular shift produced by the path of the body at the centre of the circular path per unit time.

$\therefore$ $$\text{Angular velocity} = \frac{\text{angular displacement}}{\text{time}}$$

or, $$\omega = \frac{\theta}{t} \tag{5.3}$$

5.1.3 Angular Acceleration

It represents the rate of variation of the angular velocity of a body moving in a fixed circular path.

$\therefore$ $$\text{Angular acceleration} = \frac{\text{angular velocity}}{\text{time}}$$

or, $$\alpha = \frac{\omega}{t} \tag{5.4}$$

EXAMPLE 5.1

Example 5.1 The angular velocity of a wheel is 300 rpm. Calculate (i) time period, (ii) frequency and (iii) time for angular displacement of 60º.

Solution

(i) Angular velocity $\omega = 2\pi n = 2\pi \times 5 = 10\pi\,\text{rad/s}$ ($n = 5$ rps).

So, time period, $T = \frac{2\pi}{\omega} = \frac{2\pi}{10\pi} = 0.2\text{ s}$.

EXAMPLE 5.1

(ii) Frequency, $n = \frac{1}{T} = 5\text{Hz}.$

(iii) Angular displacement = 60º = $\frac{\pi}{3}$ rad.

Now $\theta = \omega t$

$\therefore$ $t = \frac{\theta}{\omega} = \frac{\pi/3}{10\pi} = \frac{1}{30}$ s , which is the required time.

EXAMPLE 5.2

Example 5.2 For a clock it shows that the length of the minute hand as 50 cm. Calculate its angular velocity and linear velocity?

Solution

Its angular velocity,

$$\omega = \frac{2\pi}{3600} = \frac{\pi}{1800} \text{ rad / s.}$$

and the linear velocity of the tip of the minute hand is

$$v = r\omega = 0.5 \times \frac{\pi}{1800} = 87 \times 10^{-4}\,\text{ms}^{-1}.$$

5.2 CENTRIPETAL ACCELERATION

When a particle moves along a circular path with uniform angular velocity (ω) and hence with uniform linear velocity (v), the motion is said to be uniform circular motion. When a body is moving in a circular path with uniform linear velocity then the acceleration acting to the centre of its motion is known as the centripetal acceleration. As the velocity of the particle is acting towards the tangent of the circular path and the acceleration is directed towards the centre of the path so they are perpendicular to each other.

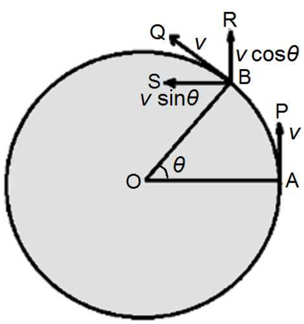

Fig. 5.2: Centripetal acceleration

Let m be the mass of a particle rotating in a circular orbit with radius r and linear velocity v. Let, the particle has an angular displacement θ when it goes from its initial position A to its final position B (Fig. 5.2).

$\therefore$ The particle has an angular velocity, $\omega = \frac{\theta}{t}$.

Now, the velocity of the particle at point A is acting towards the tangent AP and the velocity of the particle at point B is acting towards the tangent BQ with two components; one ($v\cos\theta$) along BR and the other ($v\sin\theta$) along BS.

If θ is small then, $\sin\theta \to \theta$ and $\cos\theta \to 1$.

Now, the velocity of the particle along AP is initially v and finally $v\cos\theta = v \times 1 = v$.

$\therefore$ Change in velocity = $(v - v) = 0$.

Thus, the acceleration (f) $= \dfrac{\text{change in velocity}}{\text{time}} = 0.$

Again, initial velocity of the body along AO is 0 and the final velocity along AO is $v\sin\theta = v \times \theta = v\theta.$

$\therefore$ Change in velocity will be $\dfrac{v\theta - 0}{t} = \dfrac{v\theta}{t} = v\omega = r\omega \times \omega$

$$= \omega^2 r = \left(\frac{v}{r}\right)^2 \times r = \frac{v^2}{r}$$

and this is acting to the centre of the motion. Thus, the centripetal acceleration acting towards the centre of the circular path can be represented as,

$$f_C = \omega^2 r = \frac{v^2}{r} \tag{5.5}$$

5.3 CENTRIPETAL FORCE

It is the force required to maintain the circular motion of a body in fixed circular path acting along the centre of the path and perpendicular to the velocity of the body.

Now, we know for uniform circular motion the centripetal acceleration acting on a body of mass m is,

$$f_C = \omega^2 r = \frac{v^2}{r}$$

Thus, the centripetal force acting on the body is $= mf_C = m\dfrac{v^2}{r} = m\omega^2 r$

$$\therefore \qquad F_C = m\frac{v^2}{r} = m\omega^2 r \tag{5.6}$$

SCAN ME

for viewing circular motion and centripetal acceleration as a limit

Examples: (i) When a piece of stone is tied with a rope and is rotated then the stone experiences a force of tension towards the centre and this is due to the centripetal force acting on the piece of stone.

(ii) In case of revolution of any planet around the Sun the gravitational force on the planet by the Sun gives the required centripetal force.

Example 5.3 A body of mass 100 g is uniformly rotating in a circular orbit of radius 50 cm with angular speed of 60 rpm. Calculate the centripetal force acting on it.

Solution

Here, $\omega = 2\pi \times \dfrac{60}{60} = 2\pi \text{ rad/s}$, $r = 50$ cm and $m = 100$ g.

Thus, the centripetal force acting on the body is

$$F_C = m\omega^2 r = 100 \times (2\pi)^2 \times 50 \text{ dyne} = 1.97 \text{ N}.$$

EXAMPLE 5.3

5.4 CENTRIFUGAL REACTION AND CENTRIFUGAL FORCE

When a particle is moving in a circular path then due to the centripetal force acting towards the centre of the path by some external agent, the body exerts an outward force equal to the centripetal force along the opposite direction and this is called the centrifugal reaction.

In Fig. 5.3, F_c is the centripetal force and F_R is the force due to centrifugal reaction acting in the outward direction from the centre of the path.

However, the centripetal force is acting on the body moving in the circular path while that of the centrifugal reaction will act on the agent which causes the motion of the body. Thus centrifugal reaction is basically the reaction force to the centripetal force.

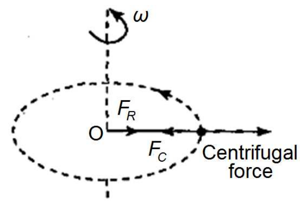

Fig. 5.3: Concept of centrifugal reaction

Although this force is not an actual force but it is a pseudo force as it can only be experienced in an accelerated frame of reference and not in an inertial reference frame.

Centrifugal force: Whenever a body is moving in a circular path with an uniform angular velocity in an accelerated (hence non-inertial) reference frame, then a force is acting on it in the radially outward direction having magnitude equal to that of the centripetal force acting on the body. This fictitious force which can only be experienced in an accelerated frame of reference as a reaction force of the centripetal force is termed as the centrifugal force and is given by,

$$F_g = m\frac{v^2}{r} = m\omega^2 r \tag{5.7}$$

5.5 MOMENT OF FORCE

When a rigid body is performing rotational motion about an axis of rotation with the application of a force then one can measure the turning effect of the applied force on the body by torque. It is basically the analogous rotational force in translational motion.

Definition: Moment of force or torque about a fixed axis of rotation may be defined as the turning effect produced by the force and is measured by product of the force and perpendicular distance from the rotational axis to the line of force acting. It is acting normal to the plane containing the line of force acting and the position vector.

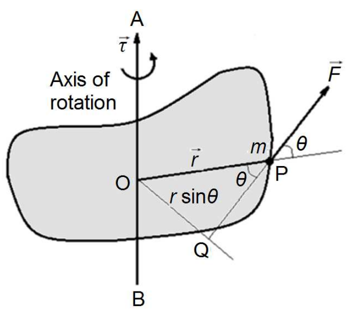

Fig. 5.4: Moment of force

Let, a force $\vec{F}$ is applied on a rigid body at a point P and also let $\vec{r}$ be the position vector of the point of application of the force with respect to the point O on the axis of rotation (Fig. 5.4).

Now, the moment of force or the torque about the point O is,

$$\tau = OQ.F$$

or,

$$\tau = r\sin\theta .F \tag{5.8}$$

where θ is the angle between $\vec{r}$ and $\vec{F}$.

In vector notation we can write,

$$\vec{\tau} = \vec{r}\times\vec{F} \tag{5.9}$$

Moment of force or torque is a vector quantity.

Direction or sign of torque: If due to the torque the rotation of the body goes clockwise, the moment is considered negative while for anti clockwise rotation, the moment is taken as positive and it is perpendicular to the plane of the paper.

Principle of moments: At equilibrium of a body the sum of all anticlockwise moments (positive moment) should be equal to the sum of all clockwise moments (negative moment) and this is known as the *principle of moments* at equilibrium.

5.5.1 Dimension and Units of Torque

Dimension of torque is $[\tau]=[r\sin\theta . F]=\left[L.MLT^{-2}\right]$

$$=\left[ML^2T^{-2}\right]$$

So, the dimension of torque is the same as that of work or energy.

SI unit of torque is N-m and CGS unit of torque is dyne-cm.

5.6 COUPLE AND MOMENT OF A COUPLE

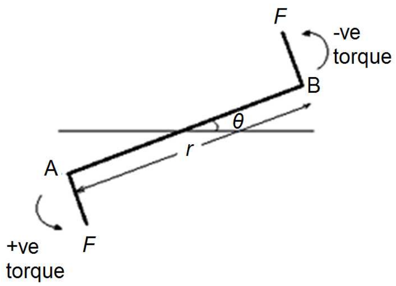

Fig. 5.5: Moment of couple

If two equal and unlike parallel forces act on a body at a certain distance at two different points then they will constitute a couple and produce similar turning effects like that in torque. The moment of the couple (Fig. 5.5) about any point in its plane is equal to the product of one force (F) forming the couple and the perpendicular distance (r) between the forces. It is given by,

$$\tau = rF\sin\theta$$

and in vector form it is, $\vec{\tau}=\vec{r}\times\vec{F}$

It possesses the same units and dimensions as that of torque.

5.7 MOMENT OF INERTIA

It represents a quantity to express the movement of a body for resisting angular acceleration.

This is obtained by the algebraic sum of the products of the mass of individual particle and the square of the distance from rotational axis.

Let us consider a rigid body which is under rotation about the axis AB with a constant angular velocity ω as shown in Fig. 5.6. So, the body must possess some rotational kinetic energy.

Now, the linear velocity of the particle of mass m_1 is $v_1 = r_1\omega$ and so its kinetic energy is, $\frac{1}{2}m_1v_1^2=\frac{1}{2}m_1r_1^2\omega^2$.

So, the total kinetic energy of the whole body is,

$$E = E_1 + E_2 + \ldots\ldots +E_n$$

$$=\frac{1}{2}\omega^2\left(m_1r_1^2+m_2r_2^2+\ldots.+m_nr_n^2\right)=\frac{1}{2}\omega^2\sum_{i=1}^{n}m_ir_i^2$$

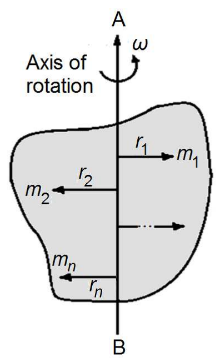

Fig. 5.6: Moment of inertia

or, $$E = \frac{1}{2} I\omega^2 \tag{5.10}$$

where $I = \sum_{i=1}^{n} m_i r_i^2$ is the moment of inertia (MI) of the body about the axis of rotation.

$\therefore$ $$I = \sum_{i=1}^{n} m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2 \tag{5.11}$$

If we compare Eq. (5.11) with the expression for kinetic energy in translational motion which is $E = \frac{1}{2} mv^2$, then we can say that MI is the rotational analogue of mass in translational motion and plays a very important role in rotational mechanics.

Definition: MI of a body about an axis, as already mentioned, may be defined as the sum of the products of masses of all the constituent particles of the body and squares of their respective perpendicular distances from the axis of rotation.

Thus, $$I = m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2 = \sum_{i=1}^{n} m_i r_i^2$$

or, in general for a body of mass M rotating around a particular axis at a distance R from it, the MI about the axis of rotation is

$$I = MR^2 \tag{5.12}$$

From Eqs. (5.11) and (5.12); we can say that the MI of a body depends on its —

(i) mass,
(ii) rotational axis,
(iii) shape,
(iv) mass distribution and
(v) distance from the rotational axis.

Physical significance of moment of inertia: MI is basically the rotational analogue of mass in translational motion and so it is a measure of inertia of the body.

5.7.1 Relation between Torque and Moment of Inertia

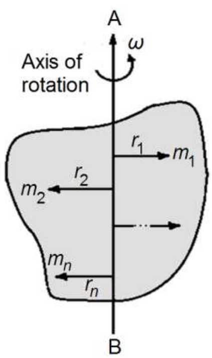

Fig. 5.7: Relationship between torque and MI

Let us consider that a rigid body is moving about the axis AB having constant angular velocity ω as shown in Fig. 5.7. Now, the moment of force acting on the body of mass m_1, about AB is,

$$\tau_1 = m_1 f_1 r_1 = m_1 r_1^2 \alpha$$

So, the total moment of force about the axis of rotation for the whole body is,

$$\tau = \tau_1 + \tau_2 + ... + \tau_n = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + ... + m_n r_n^2 \alpha$$

or, $$\tau = \left(m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2\right)\alpha = \left(\sum_{i=1}^{n} m_i r_i^2\right)\alpha$$

or, $$\tau = I\alpha \tag{5.13}$$

So, Torque = MI × Angular acceleration.

5.8 ANGULAR MOMENTUM

As we are already aware of the fact that the angular momentum of a body is the rotational analogue of linear momentum in translational motion, as like torque which is the moment of force, angular momentum can be defined as the moment of linear momentum of a body.

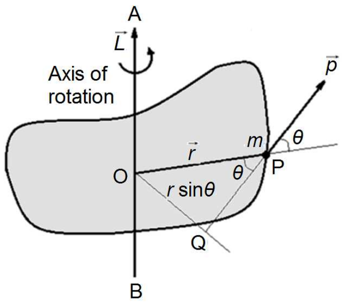

Fig. 5.8: Angular momentum of a rigid body

Definition: For a body, the angular momentum about an axis represents moment of the linear momentum about the axis. It is therefore the product of magnitude of its linear momentum and the perpendicular distance to its line of motion from the given axis.

Let $\vec{p}$ represents the linear momentum of a particle of mass m situated in a system of particles moving with angular velocity ω and $\vec{r}$ be the position vector of the particle with respect to a given point O (Fig. 5.8) on the axis of rotation. Then the angular momentum of the particle about the point O will be,

$$L = pr\sin\theta = mvr\sin\theta \tag{5.14}$$

where θ is the angle between $\vec{r}$ and $\vec{v}$ and m is the particle mass.

In vector notation we have,

$$\vec{L} = \vec{r} \times \vec{p} \tag{5.15}$$

Angular momentum is a vector quantity and is normal to the plane of the paper and thus it becomes perpendicular to both $\vec{p}$ and $\vec{r}$.

Dimension and units of angular momentum: Dimension of angular momentum is = $[ML^2T^{-1}]$,

SI unit of angular momentum is kgm^2s^{-2}

and CGS unit of angular momentum is $g\text{-}cm^2s^{-2}$.

5.8.1 Relation between Angular Momentum and Moment of Inertia

We assume that a rigid body is rotating about the axis AB with a constant angular velocity ω as shown in Fig. 5.9.

Now, the angular momentum of the body of mass m_1 about AB is,

$$L_1 = p_1r_1 = m_1v_1r_1 = m_1(r_1\omega)r_1 = m_1r_1^2\omega$$

So, the total angular momentum of the whole body about the axis of rotation AB is,

$$L = L_1 + L_2 + \ldots + L_n = m_1r_1^2\omega + m_2r_2^2\omega + \ldots + m_nr_n^2\omega$$

or,

$$L = \left(m_1r_1^2 + m_2r_2^2 + \ldots + m_nr_n^2\right)\omega = \left(\sum_{i=1}^{n} m_ir_i^2\right)\omega$$

or,

$$L = I\omega \tag{5.16}$$

So, Angular momentum = MI × Angular velocity.

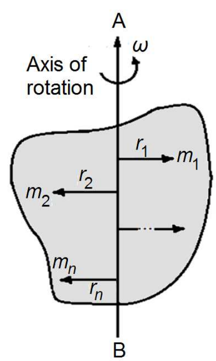

Fig. 5.9: Relationship between angular momentum and MI

5.8.2 Relationship between Torque and Angular Momentum

We know,

angular momentum $L = I\omega$ (a)

and torque $\tau = I\alpha$ (b)

Now, differentiating equation (a) we get,

$$\frac{dL}{dt} = I\frac{d\omega}{dt} = I\alpha \quad \text{(c)}$$

So, from equation (b) and (c) we get,

$$\tau = \frac{dL}{dt} \quad (5.17)$$

which gives the relationship between angular momentum and torque and states that the rate of change of angular momentum is the external torque.

5.8.3 Conservation of Angular Momentum

We know the relationship between angular momentum and external torque is, $\tau = \frac{dL}{dt}$

Now, if $\tau = 0$ then $\frac{dL}{dt} = 0$

i.e., $L = \text{constant}$

or, $I\omega = \text{constant}$

i.e., $I_1\omega_1 = I_2\omega_2$ (5.18)

So, the angular momentum of a body remains conserved when no external torque is applied. This is known as the conservation law of angular momentum.

5.9 THEOREMS OF MI

In many cases the MI can be calculated conveniently by applying the following two important theorems.

5.9.1 Parallel Axis Theorem

The MI of a body about an axis is same as its MI (I_0) about a parallel axes through its centre of mass (CM) plus the multiplication of the mass (M) of the body under consideration and the distance square between the two axes. Mathematically we can express as,

$$I = I_0 + Ma^2 \quad (5.19)$$

where a is the distance between the parallel axes.

5.9.2 Perpendicular Axis Theorem

The sum of MI of a plane laminar body, about any two mutually perpendicular axes in its plane, is same as the MI of the body about a third axis perpendicular to the body and passing through the intersection of the first two axes.

According to this theorem if I_x and I_y represents the MI of a body about X and Y axis respectively, then its MI about the Z axis which is evidently perpendicular to both the X and Y axis can be represented as,

$$I_z = I_x + I_y \tag{5.20}$$

5.10 RADIUS OF GYRATION

Independent of the shape of the body, we can observe a point always where if the total mass of the body were supposed to be concentrated, the kinetic energy of rotation remains the same as that of the body itself about the same axis. The distance K of that point from the rotational axis is termed as the radius of gyration of the body about that axis.

If M be the total mass and K the radius of gyration of the body, then we have

$$(K.E.)_{\text{rot}} = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\sum mx^2\right)\omega^2 = \frac{1}{2}MK^2\omega^2 \tag{5.21}$$

$\therefore$ $$MK^2 = \sum mx^2 = I$$

or, $$K = \sqrt{\frac{I}{M}} \tag{5.22 (a)}$$

We may also write $MK^2 = \sum mx^2 = mn\left(\dfrac{x_1^2 + x_2^2 + \ldots\ldots}{n}\right)$

where n is the number of particles, each of mass m, into which the given mass M may be divided so that $M = mn$.

$\therefore$ $$K^2 = \frac{x_1^2 + x_2^2 + \ldots\ldots}{n}$$

or, $$K = \sqrt{\frac{\sum x_i^2}{n}} \tag{5.22 (b)}$$

Hence, the radius of gyration represents the square root of the average of the square distance of the particles constituting of the body from the rotational axis. Radius of gyration about an axis is looked upon as the radius of a thin uniform ring with the centre on the axis and having the same mass as the given body and the same MI. Accordingly, the radius of gyration has the dimension of length [L].

5.11 MOMENT OF INERTIA FOR SYMMETRIC BODIES

Let us calculate the MI for simple symmetric systems.

Case 1: About an axis passing through the centre of mass (CM) and perpendicular to a straight uniform rod

Let a mass element dm of length dx be located at a distance x from the CM of the rod. MI of the rod about the axis AB (Fig. 5.10) will be,

$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 dm = \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \left(\frac{m}{l}\right) dx \qquad \left(\text{mass per unit length} = \frac{m}{l}\right)$$

$$= \frac{m}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 dx = \frac{m}{l}\left[\frac{x^3}{3}\right]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$= \frac{m}{3l}\left[\frac{l^3}{8} - \left(-\frac{l^3}{8}\right)\right] = \frac{ml^3}{12l} = \frac{ml^2}{12} \qquad (5.23)$$

$\therefore$ Radius of gyration, $K = \frac{l}{\sqrt{12}}$ (5.24)

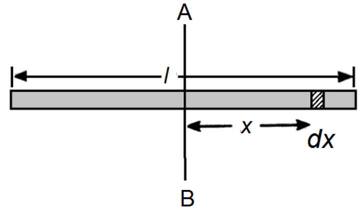

Fig. 5.10: Axis passing through CM of the rod

Case 2: About an axis passing through one end and perpendicular to a straight uniform rod

In this case (Fig. 5.11), the limits of integration will be 0 to l, instead of $-\frac{l}{2}$ to $\frac{l}{2}$. So, MI about AB is,

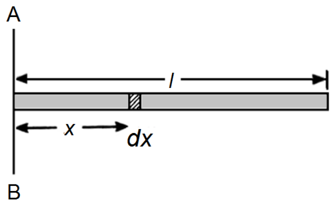

Fig. 5.11: Axis passing through one end of the rod

$$I = \int_0^l x^2 dm = \int_0^l x^2\left(\frac{m}{l}\right)dx$$

$$= \frac{m}{l}\int_0^l x^2 dx = \frac{m}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{ml^2}{3} \qquad (5.25)$$

$\therefore$ Radius of gyration, $K = \frac{l}{\sqrt{3}}$ (5.26)

Case 3: About an axis perpendicular to a uniform circular disc

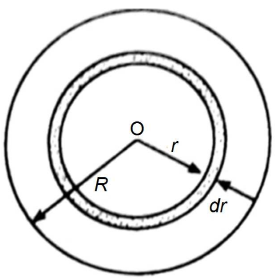

Fig. 5.12: Axis passing through the centre and perpendicular to the plane of the disc

Let the disc be divided into concentric circular rings [Fig. 5.12]. Choose the ring of width dr and mass element dm at a distance r from the centre of the disc.

$\therefore$ dm = area of the ring × mass per unit area $= 2\pi r dr \times \frac{m}{\pi r^2}$

The MI of the ring under reference about the axis is

$$dI = r^2 dm$$

So, the MI of the disc about O is

$$I = \int dI = \int r^2 dm = \frac{2m}{R^2}\int_0^R r^3 dr = \frac{2m}{R^2}.\frac{R^4}{4}$$

$\therefore$ Moment of inertia, $I = \frac{1}{2}mR^2$ (5.27)

and radius of gyration, $K = \frac{R}{\sqrt{2}}$ (5.28)

Case 4: About an axis passing through CM and perpendicular to the plane of an annular ring

The procedure would be identical with that of previous case, with the following modifications: The limits of integration will be R_1 and R_2, and mass element $dm = (2\pi r dr)\times\left[\frac{m}{\pi\left(R_2^2 - R_1^2\right)}\right]$.

$\therefore$ The MI of the annular ring [Fig. 5.13] about the ring axis is

$$I = \int_{R_1}^{R_2} \frac{2m}{R_2^2 - R_1^2} r^3 dr$$

$$I = \frac{2m}{R_2^2 - R_1^2} \int_{R_1}^{R_2} r^3 dr = \frac{2m}{R_2^2 - R_1^2} \left[\frac{r^4}{4} \right]_{R_1}^{R_2}$$

$$= \frac{2m}{R_2^2 - R_1^2} \frac{R_2^4 - R_1^4}{4} = \frac{1}{2} m \left(R_1^2 + R_2^2 \right) \qquad (5.29)$$

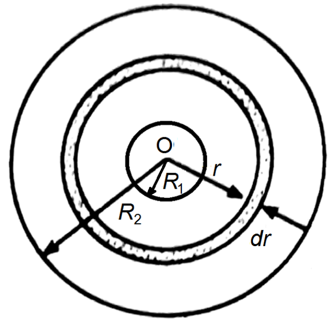

Fig. 5.13: Axis passing through the centre and perpendicular to the plane of the annular ring

$\therefore$ Radius of gyration,

$$K = \sqrt{\frac{\left(R_1^2 + R_2^2 \right)}{2}} \qquad (5.30)$$

Case 5: About the axis of a uniform cylinder

Let the cylinder be divided into thin discs perpendicular to the axis. The MI of any of the discs is by Case 3 above is $\frac{1}{2} mR^2$, where m is the mass of the disc.

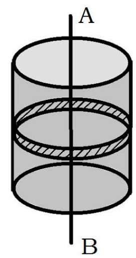

Fig. 5.14: About the axis of the cylinder

So, the MI of the whole cylinder is

$$I = \sum \frac{1}{2} mR^2 = \frac{1}{2} R^2 \sum m = \frac{1}{2} MR^2 \qquad (5.31)$$

$$\therefore \text{ Radius of gyration, } K = \frac{R}{\sqrt{2}} \qquad (5.32)$$

Case 6: About the axis of a uniform cylindrical shell

Let the cylindrical shell be divided into annular rings, perpendicular to its axis. The MI of any of the rings is, by Case 4 above, $\frac{1}{2} m \left(R_2^2 + R_1^2 \right)$.

So, the MI of the tube is

$$I = \sum \frac{1}{2} m \left(R_1^2 + R_2^2 \right) = \frac{1}{2} \left(R_1^2 + R_2^2 \right) \sum m = \frac{1}{2} M \left(R_1^2 + R_2^2 \right) \qquad (5.33)$$

$$\therefore \text{ Radius of gyration, } K = \sqrt{\frac{\left(R_1^2 + R_2^2 \right)}{2}} \qquad (5.34)$$

Case 7: About the CM and parallel to one side of a uniform rectangular lamina

Let a and b be the sides of the rectangular lamina [Fig. 5.15] and let the axis be parallel to the side b. Let the lamina be divided into thin strips, each of width dx, parallel to side b. The MI of one such strip, distant x from the axis is

$$dI = \frac{m}{a} x^2 dx \qquad (\because \text{ mass per unit area } = m/ab \text{, area of strip} = bdx)$$

The MI of the whole lamina about the said axis is

$$I = \int dI = \int_{-a/2}^{a/2} \frac{m}{a} x^2 dx = \frac{m}{a}\left[\frac{x^3}{3}\right]_{-a/2}^{a/2} = \frac{1}{12} ma^2 \tag{5.35}$$

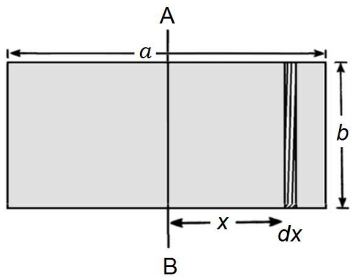

Fig. 5.15: Axis through the CM and parallel to one side of a uniform rectangular lamina

$\therefore$ Radius of gyration, $K = \dfrac{a}{\sqrt{12}}$ (5.36)

If the axis were parallel to the side a, then we should have

$$I = \frac{1}{12} mb^2$$

and $$K = \frac{b}{\sqrt{12}}.$$

Case 8: About a diameter of a uniform solid sphere

Let the sphere be divided into concentric spherical shells. Choose one such shell of radius r and thickness dr.

Consider a thin circular slice of width dx at a distance x from the center of the solid sphere (Fig. 5.16). Mass of this element is, $dm = \pi\left(r^2 - x^2\right) dx \rho$, where ρ is the density of the material.

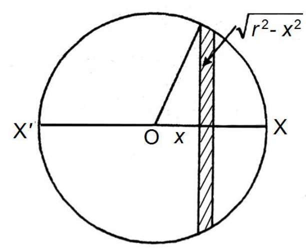

Fig. 5.16: About a diameter of a uniform solid sphere

So, its MI about the centre O is

$$dI = \frac{1}{2}\pi\left(r^2 - x^2\right) dx \rho \left(\sqrt{r^2 - x^2}\right)^2$$

$$= \frac{\pi\rho}{2}\left(r^2 - x^2\right)^2 dx$$

The MI of the solid sphere about the diameter XOX' is

$$I = \int_{-r}^{r} \frac{\pi\rho}{2}\left(r^2 - x^2\right)^2 dx$$

or, $$I = \frac{\pi}{2}\left(\frac{3m}{4\pi r^3}\right)\int_{-r}^{r}\left(r^2 - x^2\right)^2 dx = \frac{2}{5} mr^2 \quad \left(\because m = \frac{4}{3}\pi\rho r^3\right) \tag{5.37}$$

SCAN ME

for simulation of MI

$\therefore$ Radius of gyration, $K = \sqrt{\dfrac{2}{5}}\, r$ (5.38)

EXAMPLE 5.4

Example 5.4 If the radius of the earth shrinks to 1/4th of its previous value, calculate the length of a day.

Solution

From the conservation of angular momentum we have, $I_1\omega_1 = I_2\omega_2$

where $I_1 = \dfrac{2}{5} MR^2$ and $I_2 = \dfrac{2}{5} M\left(\dfrac{R}{4}\right)^2 = \dfrac{1}{40} MR^2$.

Thus, $\frac{2}{5}MR^2 \times \omega_1 = \frac{1}{40}MR^2 \times \omega_2$

or, $\frac{2\omega_1}{5} = \frac{\omega_2}{40}$

or, $\omega_2 = 16\omega_1 = 16 \times \frac{2\pi}{24} = \frac{4\pi}{3}$ rad/hr.

or, $T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{\frac{4\pi}{3}} = 1.5\,\text{hr}.$

So, the length of a day is 1.5 hours.

EXAMPLE 5.4

Example 5.5 Show that the kinetic energy of a thin rod of mass m per unit length having length l rotating about an axis through the midpoint and perpendicular to its length with angular velocity ω is $\frac{1}{24}m\omega^2 l^2$.

Solution

The moment of inertia of the rod about the given axis is

$$I = \frac{1}{12}Ml^2 = \frac{1}{12}ml.l^2 = \frac{1}{12}ml^3 \qquad [\because M = ml]$$

So, kinetic energy of rotation will be,

$$\frac{1}{2}I\omega^2 = \frac{1}{24}ml^3\omega^2.$$

EXAMPLE 5.5

Example 5.6 Calculate the moment of inertia for a thin circular ring of mass 150 g and radius 50 cm about an axis in the plane of the ring when the axis (a) passes through the centre and (b) is 10 cm away from the centre.

Solution

(a) The moment of inertia of the ring is to be taken, plainly about the diameter. The required moment of inertia,

$$I = \frac{1}{2}mr^2 = \frac{1}{2} \times 150 \times 50^2 = 1.875 \times 10^5\,\text{g/cm}^2.$$

(b) The moment of inertia of the ring can be calculated by using parallel axis theorem as $I = \frac{1}{2}mr^2 + mx^2$

$$= \frac{1}{2} \times 150 \times 50^2 + 150 \times 10^2 = 2.025 \times 10^5\,\text{g/cm}^2.$$

EXAMPLE 5.6

5.12 EULER'S LAWS OF MOTION

These are equation of motion which extends Newton's laws of motion for a point mass to rigid body motion. In a deformable body the internal force distribution is not necessarily uniform all through the body leading to a variation of stress distribution from point to point. This variation is governed by Newton's second law of motion and can be extended to a body of continuous mass distribution. In case of bodies with continuous mass distribution these laws are known as the Euler's laws of motion after his formulation. The laws are stated below:

First law: It can be stated in the following way:

For a body the linear momentum is same as the product of the mass of the body and the velocity, taking into account the centre of mass.

Mathematically it can be expressed as,

$$\vec{p} = m\vec{v}_{cm} \tag{5.39}$$

The law can alternately be expressed as,

$$\vec{F} = m\vec{f}_{cm} \tag{5.40}$$

by simply taking derivative on both sides of the equation.

Second law: It states that the time rate of change of angular momentum about a fixed point in an inertial frame of reference is equal to the sum of the external moments of force (or torque) acting on that body about that point.

Mathematically we can write it as,

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{5.41}$$

For, a body translating and rotating in only two dimensions it can be expressed as,

$$\vec{\tau} = m\vec{r}_{cm} \times \vec{f}_{cm} + I\vec{\alpha} \tag{5.42}$$

where, $\vec{r}_{cm}$ is the position of the CM with respect to the point about which moments are calculated.

I and $\vec{\alpha}$ are respectively the MI and angular acceleration of the body about its CM.

UNIT SUMMARY

- ***Rotational motion***

  Angular displacement $S = r\,\theta$

  Angular velocity $\omega = \dfrac{\theta}{t}$

  Angular acceleration $\alpha = \dfrac{\omega}{t}$

- ***Centripetal acceleration***

  $$f_C = \omega^2 r = \frac{v^2}{r}$$

- ***Centripetal force***

  $$F_C = \frac{m v^2}{r} = m\,\omega^2 r$$

- ***Centrifugal reaction and centrifugal force***

  $$F_g = \frac{m v^2}{r} = m\,\omega^2 r$$

- ***Moment of force (torque)***

  $$\vec{\tau} = \vec{r} \times \vec{F}$$

- ***Moment of inertia***

  $$I = m_1 r_1^2 + m_2 r_2^2 + \dots + m_n r_n^2 = \sum_{i=1}^{n} m_i r_i^2$$

- ***Relationship between torque and moment of inertia***

  $$\tau = I\alpha$$

- ***Angular momentum***

  $$\vec{L} = \vec{r} \times \vec{p}$$

- ***Relationship between angular momentum and moment of inertia***

  $$L = I\omega$$

- ***Relationship between torque and angular momentum***

  $$\tau = \frac{dL}{dt}$$

- ***Conservation of angular momentum***

  $$I_1\omega_1 = I_2\omega_2$$

- ***Parallel axis theorem***

  $$I = I_0 + Ma^2$$

- ***Perpendicular axis theorem***

  $$I_z = I_x + I_y$$

- ***Radius of gyration***

$$K = \sqrt{\frac{\sum x_i^2}{n}}$$

- ***Moment of inertia calculations for some regular symmetric bodies***

About an axis passing through CM and perpendicular to a uniform rod

$$I = \frac{ml^2}{12} \qquad K = \frac{l}{\sqrt{12}}$$

About an axis passing through one end and perpendicular to a straight rod

$$I = \frac{ml^2}{3} \qquad K = \frac{l}{\sqrt{3}}$$

About an axis perpendicular to a uniform circular disc

$$I = \frac{1}{2}mR^2 \qquad K = \frac{R}{\sqrt{2}}$$

About the CM and perpendicular to the plane of an annular ring

$$I = \frac{1}{2}m\left(R_1^2 + R_2^2\right) \qquad K = \sqrt{\frac{\left(R_1^2 + R_2^2\right)}{2}}$$

About the axis of a uniform cylinder

$$I = \frac{1}{2}mR^2 \qquad K = \frac{R}{\sqrt{2}}$$

About the axis of a uniform cylindrical shell

$$I = \frac{1}{2}m\left(R_1^2 + R_2^2\right) \qquad K = \sqrt{\frac{\left(R_1^2 + R_2^2\right)}{2}}$$

About the CM and parallel to one side of a uniform rectangular lamina

$$I = \frac{1}{12}ma^2 \qquad K = \frac{a}{\sqrt{12}}$$

$$I = \frac{1}{12}mb^2 \qquad K = \frac{b}{\sqrt{12}}$$

About a diameter of a uniform solid sphere

$$I = \frac{2}{5}mr^2 \qquad K = \sqrt{\frac{2}{5}}r$$

- ***Euler's laws of motion***

First law

$$\vec{p} = m\vec{v}_{cm} \qquad \vec{F} = m\vec{f}_{cm}$$

Second law

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$\vec{\tau} = m\vec{r}_{cm} \times \vec{f}_{cm} + I\vec{\alpha}$$

EXERCISES

Multiple Choice Questions

5.1 The relation between angular momentum L, moment of inertia I and angular velocity ω is

(a) $L = \frac{I}{\omega}$ (b) $L = I^2\omega$ (c) $L = \frac{I^2}{\omega}$ (d) $L = I\omega$

5.2 The dimensional formula of angular momentum is

(a) ML^2T^{-2} (b) ML^2T^{-1} (c) MLT^{-2} (d) MLT^{-1}

5.3 The magnitude of centripetal force is

(a) mv^2r (b) $\frac{mv^2}{r}$ (c) mv^2r^2 (d) $\frac{mv}{r}$

5.4 When milk is charred; cream separates out because of

(a) gravitational force (b) viscous force (c) centrifugal force (d) mechanical force

5.5 The dimensions of angular acceleration are

(a) $[LT^{-2}]$ (b) $[LT^{-1}]$ (c) $[T^{-3}]$ (d) $[T^{-2}]$

5.6 Centripetal force acts on

(a) the body executing circular motion

(b) the external agent which causes the angular motion

(c) both the body and the external agent (d) none of these

5.7 The relation between torque τ, moment of inertia I and angular acceleration α is

(a) $\tau = I^2\alpha$ (b) $\tau = \frac{I^2}{\alpha}$ (c) $\tau = \frac{I}{\alpha}$ (d) $\tau = I\alpha$

5.8 Centrifugal reaction acts on

(a) the body executing circular motion

(b) the external agent which causes the circular motion

(c) both the body and the external agent (d) none of these

5.9 The SI unit of torque is

(a) Nm^2 (b) Nm (c) Nm^{-2} (d) Nm^{-1}

5.10 The rotational analogue of force is

(a) torque (b) moment of inertia (c) angular momentum (d) angular velocity

5.11 Rate of change of angular momentum of a particle is equal to

(a) product of moment of inertia and angular velocity

(b) product of torque and angular velocity

(c) torque acting on it (d) kinetic energy of rotation

5.12 A mass m is moving with a constant velocity parallel to X-axis. Its angular momentum is

(a) zero (b) remains constant (c) goes on increasing (d) goes on decreasing

5.13 The kinetic energy of rotation of a body of mass m, angular velocity ω and moment of inertia I is given by,

(a) $\frac{1}{2}(I\omega)^2$ (b) $\frac{1}{2}I\omega^2$ (c) $I\omega$ (d) $I^2\omega$

5.14 The angular moment of a particle of mass m, velocity v, momentum p at position r (with respect to the origin) is given by

(a) mvr (b) $r \times p$ (c) $v \times p$ (d) mv / r

5.15 A particle moves in a circular path of radius r with a constant angular velocity ω. Its linear velocity is

(a) $r\omega$ (b) r / ω (c) ω / r (d) $r\omega^2$

5.16 The centripetal force is given by

(a) $\frac{mv^2}{r}$ (b) $\frac{v^2}{r}$ (c) $mr^2\omega^2$ (d) mgr

5.17 When water in a bucket is whirled fast, overhead water does not fall out at the top of the motion because

(a) the centripetal force on the water is greater than the weight of water

(b) the centripetal force on the water is less than the weight of water

(c) atmospheric pressure counteracts the weight

(d) centrifugal force on water is greater than the weight of water

5.18 Dimensions of angular velocity are

(a) zero (b) $[LT^{-1}]$ (c) $[T^{-1}]$ (d) $[L^{-1}]$

5.19 For a particle moving uniformly around a circular path of radius r (v = linear velocity, ω = angular velocity)

(a) $\omega = \frac{v}{r}$ (b) $\omega = vr$ (c) $\omega = \frac{r}{v}$ (d) $\omega = \frac{r^2}{v}$

5.20 Angular velocity of the second hand of a clock is

(a) 2π rad/s (b) $\frac{\pi}{30}$ rad/s (c) $\frac{\pi}{3}$ rad/s (d) $\frac{\pi}{6}$ rad/s

5.21 A particle moves with a constant velocity parallel to X axis. Its angular momentum with respect to the origin

(a) is zero (b) is constant (c) increases (d) decreases

5.22 A body is under perfect rotation. Its linear speed v is related to its angular velocity ω by $\omega = v/r$ where r is the distance of the particle from the axis. Then

(a) $\omega \propto \frac{1}{r}$ (b) $\omega \propto r$ (c) $\omega = 0$ (d) ω is independent of r

5.23 Which one must be applied to maintain the rotation of the system about a given axis?

(a) force (b) momentum (c) velocity (d) torque

5.24 A man is sitting on a rotating circular table with his arm stretched out and suddenly he then folds his arms to bring them close to the body. The angular momentum about the axis of rotation will

(a) increase (b) decrease (c) remains unchanged (d) can't be predicted

5.25 Moment of inertia of a circular wire (mass = M, radius = R) about its diameter is

(a) 1/2 MR^2 (b) 1/4 MR^2 (c) 2 MR^2 (d) MR^2

5.26 The moment of inertia of a semicircular wire (mass = M, radius = r) about a line normal to the plane of the wire through the centre is

(a) $\frac{1}{4}Mr^2$ (b) $\frac{1}{2}Mr^2$ (c) $\frac{2}{5}Mr^2$ (d) Mr^2

Answers of Multiple Choice Questions

5.1 (d), 5.2 (b), 5.3 (b), 5.4 (c), 5.5 (d), 5.6 (a), 5.7 (d), 5.8 (b), 5.9 (b), 5.10 (a), 5.11 (c), 5.12 (a), 5.13 (b), 5.14 (a), 5.15 (a), 5.16 (a), 5.17 (d), 5.18 (c), 5.19 (a), 5.20 (b), 5.21 (c), 5.22 (a), 5.23 (d), 5.24 (c), 5.25 (b), 5.26 (d)

Short and Long Answer Type Questions

Category I

5.1 Find a relation between angular velocity and angular acceleration.

5.2 Explain uniform circular motion with an illustration.

5.3 Differentiate between (i) couple and (ii) torque.

5.4 Elaborate the term moment of a force or torque with mathematical form and also mention the SI unit.

5.5 Establish the following relationships –

i) torque and angular momentum,

ii) linear and angular momentum and

iii) torque and moment of inertia.

5.6 Show the difference between centrifugal reaction and centrifugal force.

5.7 Explain moment of inertia and write down its SI unit.

5.8 Write down the expression of centripetal acceleration in terms of linear speed.

5.9 Write down the expression of centripetal force.

5.10 Explain angular momentum and hence express its SI unit.

5.11 Clearly explain the principle of conservation angular momentum.

5.12 Find a relation between torque and angular momentum.

5.13 Explain the term centripetal force acting on a rotating body.

5.14 Calculate the moment of inertia of the following objects along their axis.

i) a rod, ii) circular lamina, iii) spherical shell, iv) solid cylinder and v) solid sphere.

5.15 Calculate the angular momentum of a rigid body rotating about a fixed axis with angular velocity ω and explain the moment of inertia and radius of gyration.

5.16 Prove the theorems of parallel and perpendicular axes as applied to moment of inertia.

5.17 Calculate the moment of inertia of

a) an annular ring about an axis i) through the centre and perpendicular to its plane and ii) about a diameter.

b) a thin spherical shell about a diameter and hence of i) a solid sphere and ii) a thick shell about an axis through the centre.

Category II

5.18 Torque is related to angular acceleration. Justify the statement.

5.19 Centripetal force is a no work force. Justify.

5.20 Is moment of inertia a scalar or vector quantity? Justify your answer.

5.21 If mass of a particular body is constant, is its moment of inertia fixed?

5.22 Why centrifugal force is called fictitious force?

5.23 Why is the earth bulged at the equator and flattened at the poles?

5.24 Moment of inertia is the rotational analogues to mass of a body- explain the statement.

Numerical Problems

5.1 An electric motor drill, rated 350 W has an efficiency of 35% .What torque will it produce if it is working at 3000 *r.p.m.*? [*Ans:* 0.39 N-m]

5.2 A body of mass 50 gm is uniformly rotating in a circular orbit of radius 100 cm with angular speed of 50 rpm. Calculate the centripetal force acting on it.

5.3 If the radius of the earth shrinks to 1/2 of its previous value, calculate the length of a day.

5.4 Four spheres, each of diameter $2a$ and mass m, are placed with their centers on the four corners of a square of side b.

Calculate the moment of inertia of the system about one side of the square.

$$\left[Ans{:}\frac{2}{5}m\left(4a^2+5b^2\right)\right]$$

PRACTICALS

1. Determination of the moment of inertia of a flywheel

Theory

A flywheel is basically a heavy wheel (w) capable of rotating about the horizontal axis (AB). This is also the axis of a long narrow wheel Q which is called as the axle. To determine the MI of the flywheel about the axis AB, a cord or rope is wound on the axle several times and from the free end of the cord or rope a mass m is suspended and is allowed to fall down.

Now, as the mass is released it falls due to its own weight mg. The length of the cord or rope is so adjusted that when the mass reaches the ground the other end of the cord or rope can just leaves the axle, wheel is rotated for a known number of times n_1 such that the string is wound over n_1 turns on the axle without overlapping. Now let the mass m is at a height h from the floor. The mass is then permitted to descend down when it exerts a torque on the axle of the flywheel. By this torque the flywheel rotates making an angular acceleration. If ω represents the angular velocity of the wheel at the time when the peg detaches the axle and W is the work done against friction for a rotation, then using the law of conservation of energy,

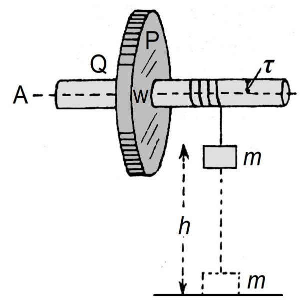

Fig. (i)

So,
$$mgh = \frac{1}{2}mr^2\omega^2 + \frac{1}{2}I\omega^2 + n_1W \quad (i)$$

Let, n_2 be the number of rotations made by the wheel before it stops. Since the kinetic energy of rotation of the flywheel is completely dissipated when it comes to rest, we can write,

$$\frac{1}{2}I\omega^2 = n_2W \quad (ii)$$

Thus,
$$mgh = \frac{1}{2}mr^2\omega^2 + \frac{1}{2}I\omega^2 + \frac{n_1}{2n_2}I\omega^2 = \frac{1}{2}mr^2\omega^2 + \frac{1}{2}I\omega^2\left(1 + \frac{n_1}{n_2}\right)$$

or,
$$I = \frac{2mgh - mr^2\omega^2}{\omega^2\left(1 + \frac{n_1}{n_2}\right)} = \frac{\frac{2mgh}{\omega^2} - mr^2}{\left(1 + \frac{n_1}{n_2}\right)} \quad (iii)$$

where h = height from the floor,

I = MI of the flywheel about the axis AB,

ω = its angular velocity when the mass is about to touch the ground,

r = radius of the axle,

m = mass suspended through string / thread,

n_1 = number of turns of string wrapped on axis and

n_2 = number of oscillation up to flywheel stopped after detaching the mass.

To measure the angular velocity ω, when the rope leaves the wheel, a suitable mark is given on the circumference of the wheel. The wheel is then allowed to continue in its rotation until it comes to rest due to friction. Let, n_2 be the number of revolutions and t is the time taken.

Then the average angular velocity of the wheel = $\frac{2\pi n_2}{t}$ rad/s.

This is equal to half of the maximum angular velocity of the wheel when the rope leaves the axle.

So, $$\omega = 2\times\frac{2\pi n_2}{t} = \frac{4\pi n_2}{t} \qquad (iv)$$

and, $$h = 2\pi r n_1 \qquad (v)$$

So, $$I = \frac{\dfrac{2mg\left(2\pi r n_1\right)t^2}{\left(4\pi n_2\right)^2} - mr^2}{\left(1+\dfrac{n_1}{n_2}\right)} = \frac{mr\left(\dfrac{gt^2}{4\pi}\dfrac{n_1}{n_2^2} - r\right)}{\left(1+\dfrac{n_1}{n_2}\right)}$$

$\therefore$ $$I = \frac{mrgt^2 n_1}{4\pi n_2^2\left(1+\dfrac{n_1}{n_2}\right)} = \frac{mrgt^2}{4\pi n_2^2\left(1+\dfrac{n_2}{n_1}\right)} \qquad \left[\text{As, } \frac{gt^2}{4\pi}\frac{n_1}{n_2^2} >> r\right] \qquad (vi)$$

Procedure

1. Determine the diameter of the axle with the help of vernier calipers for different points and then calculate the mean.
2. Attaching the mass with the flywheel, wrap the string or thread axle of flywheel for certain number of turns n_1.
3. Permit the mass to fall.
4. After fall of the mass, note the number of oscillation of flywheel (n_2) and corresponding time (t_2) till the flywheel is stopped.
5. Repeat last three steps for different masses (m) at fixed n_1.

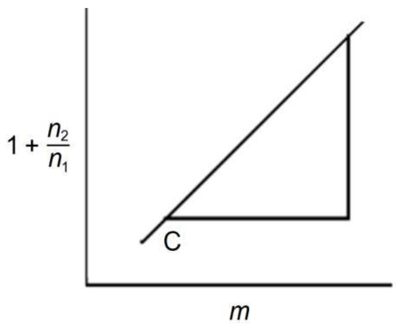

Fig. (ii)

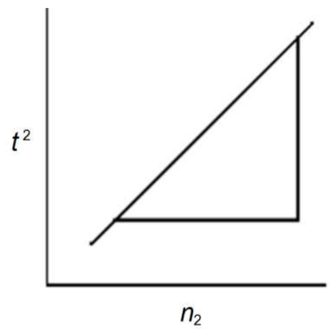

Fig. (iii)

6. Draw the graphs between mass m vs $\left(1+\dfrac{n_2}{n_1}\right)$ [Fig. (*ii*)] and between n_2 vs t^2 [Fig. (*iii*)]. The graph should be separated for each n_1.

Observation

Table 1: Determination of the radius of the axle

Least count of the vernier = ……. cm.

Sr. No.	Main scale reading	Vernier scale reading	Total reading
1			
2			
3			
4			
5			
6			

So, diameter of axle (D) = Mean of total reading = ………cm,
and hence the radius of axle ($r = D/2$) = ……… cm.

Table 2: Determination of n_2 and t

Sr. No.	n_1	n_2	m	t	$1 + n_2/ n_1$	t^2
1						
2						
3						
4						
5						
6						

Calculation

The MI can be calculated with the following formula:

$$I = \frac{mrgt^2}{4\pi n_2^2 \left(1 + \frac{n_2}{n_1}\right)}$$

Precautions

1. There should be least friction in flywheel.
2. The string length should be smaller than the height of axle from the floor.
3. No kink should be there in the string.
4. The string is to be taken thin and should be wound evenly.
5. Stop watch should be started immediately detaching the loaded string.

2. Determination of the moment of inertia of a rectangular bar

Aim

To determine the MI of a rectangular bar about an axis going through its centre and perpendicular to the plane

Apparatus

Torsional pendulum

Theory

We can determine the MI of regular or irregular bodies by torsional pendulum. Simply it is a wire W suspended from a torsion head T, which supports a cradle [Fig. (*i*)]. The whole arrangement is enclosed in a box provided with glass windows. The arrangement is so made that the torsion wire passes through the centre of gravity (CG) of the cradle. If the cradle undergoes torsional oscillation, the wire acts as the axis of oscillation. The wire is rigidly fixed at one end and at the other end the experimental body is suspended, so that the centre of mass of the body passes through the wire. When the lower end is twisted by a couple, then, within the elastic limit, the torque is proportional to the angle of twist,

i.e., $\tau \propto \theta$

or, $$\tau = C\theta \qquad (i)$$

where C = torsional constant or torque for unit angle of twist.

Now, $$\tau = I\frac{d^2\theta}{dt^2} \qquad (ii)$$

or, $$I\frac{d^2\theta}{dt^2} = C\theta$$

or, $$\frac{d^2\theta}{dt^2} = \frac{C}{I}\theta = \omega^2\theta$$

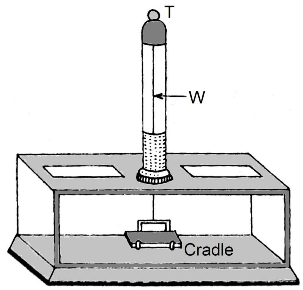

Fig. (i)

So, the angular acceleration is proportional to angular displacement.

Thus, the motion is simple harmonic. The time period of oscillation is

$$T = 2\pi\sqrt{\frac{I}{C}} \qquad (iii)\,(a)$$

a) If I_1 be the MI of the cradle and T_1 be its time period of oscillation then

$$T_1 = 2\pi\sqrt{\frac{I_1}{C}} \qquad (iii)\,(b)$$

b) Now, take a rectangular bar. Find its mass M, length l and breadth b. Let its MI is I_2 about the axis passing through the centre and perpendicular to its plane. We have,

$$I_2 = \frac{M}{12}\left(l^2 + b^2\right) \qquad (iv)$$

The bar is placed on the cradle such that the wire W passes through its centre. Its time period of oscillation T_2 is measured. Then,

$$T_2 = 2\pi\sqrt{\frac{I_1 + I_2}{C}} \qquad (v)$$

c) Next the bar is removed and the experimental bar is properly placed on the cradle. Its time period of oscillation T_3 is again measured.

So, $$T_3 = 2\pi\sqrt{\frac{I_1 + I_3}{C}} \qquad (vi)$$

Hence, from Eqs. (*iii*) (b), (*v*) and (*vi*) we get,

$$I_3 = \frac{T_3^2 - T_2^2}{T_2^2 - T_1^2} I_2 \tag{vii}$$

Procedure

1. The mass (M) of the known body is determined by a rough balance or spring balance while its breath (b) and length (l) are measured by a slide calipers and ordinary metre scale respectively. If the length of the known body is small then it should be measured by a slide calipers. Its MI I_2 is then calculated from relation (*iv*).

2. The cradle alone is made to perform torsional oscillations with small angular amplitude and the time for 30 complete oscillations is noted thrice. The mean of this three observed time when divided by 30 we get the period, T_1 of the cradle.

3. The known body is then placed horizontally on the cradle and the period of oscillation T_2 of the cradle and the known body together is determined as before.

4. The known body is then replaced by the unknown body and the period of oscillation T_3 of the cradle and the unknown body together is similarly, determined.

Observations

Mass of the bar = ……. kg.

Length of the bar = $l = \frac{l_1 + l_2 + l_3}{3}$ = ….. cm =….. m.

Breadth of the bar = $b = \frac{b_1 + b_2 + b_3}{3}$ = ….. cm =….. m.

MI of the bar about the axis passing through its centre and perpendicular to its plane is,

$$I_2 = \frac{M}{12}\left(l^2 + b^2\right) = \text{….. kg-m}^2.$$

Table 1: Determination of the periods of oscillation

Oscillating Body	Time for 30 oscillations	Mean Time (t) in s	Period T (= t/30) in s
Cradle			$= T_1$
Cradle + Rectangular bar			$= T_2$
Cradle + Unknown bar			$= T_3$

Calculations

By calculating T_1, T_2, and T_3 from the above table and using the value of I_2, we can calculate I_3 by using the formula;

$$I_3 = \frac{T_3^2 - T_2^2}{T_2^2 - T_1^2} I_2$$

Putting the values of the parameters on the right hand side, I_3 can be found out in kg-m^2.

Viva-Voce Questions

1. Mention the units for moment of inertia, torque and angular acceleration.
2. What is meant by moment of inertia? Give some examples.
3. What is radius of gyration?
4. If the applied torque to a rigid body is doubled, what will be the moment of inertia?
5. Compare the compensation of friction in this experiment for the purpose of friction compensation in Newton's second law.
6. How can you find the moment of inertia of a sphere?

References for Practicals

https://www.andrews.edu/phys/courses/p131/manual/experiment7.html
https://vlab.amrita.edu/?sub=1&brch=74&sim=571&cnt=1

KNOW MORE

By the middle of the 19th century, most people knew that Earth spins on its axis, completing a rotation once a day. However, there was no visual demonstration of the Earth's rotation, only astronomical evidence.

Rotational motion is a very common phenomenon everywhere. When we push a door, it rotates; when we pedal a bike, the wheel rotates: when we start an engine, many parts rotate; electrons rotate in an atom; H_2O molecules rotate in a microwave and the galaxies rotate in the Universe.

All motion can be classified into Translation, Rotation and Vibration. A baseball translates along a parabolic path, rotates (spins) about its centre and vibrates when it hits a bat. The earth translates around the sun in an elliptical path, rotates about its axis and vibrates during an earthquake. The translation and rotation of N_2 and O_2 molecules determines the temperature and thermodynamics of the atmosphere and hence the weather.

Activity

In order to make a curve with a bicycle or a motorcycle there must be sufficient friction between the tires and the road. This is because the frictional force gives the centripetal force that makes the bike follow the curved trajectory. But at the same time the frictional force makes the rider and the bike rotate sideways in the direction opposite to the center of the curve.

For the purpose of counteracting that rotation the rider tilts with the bike, making their weights create a tendency to rotate in the opposite direction.

This tendency to rotate is created by a force, called moment. The high speeds in motorcycle races imply higher tilt angles and to make the motorcycle tilt more the rider first turns the wheel in the opposite sense of the curve, moving his body away from the motorcycle in the direction of the center of the curve.

Interesting facts

Principles of mechanics for a particle can be extended to a system of *n* discrete material points. Some of the concepts are particularly useful in the study of Lagrange's general equations for arbitrary dynamical systems and the development of the moment of momentum principle for a system of particles gives a foundation for independent presentation of parallel results for the moment of momentum of a rigid body. Internal tensile stress produces the centripetal force which keeps a spinning object together.

A rigid body neglects the accompanying strain but if the body is not rigid this strain will make it to change shape. This can be defined as the object changing shape owing to the "centrifugal force". Celestial bodies rotating about each other frequently follow elliptic orbits. The special case of circular orbits is another example of a rotation around a fixed axis which is the line through the center of mass perpendicular to the plane of motion.

The centripetal force is provided by the gravity which is applied for a spinning celestial body. A spinning celestial body of water must require a minimum time of 3 hours and 18 minutes to rotate irrespective of size. If the fluid density is more the time can be less.

Analogy

Study of dynamics can be divided into categories; linear dynamics and rotational dynamics. Linear dynamics are associated with the objects moving in a line and covers topics like force, mass, inertia, displacement, velocity, acceleration and momentum. Rotational dynamics, on the other hand, are rotating or moving in a curved path and covers topics like torque, moment of inertia, rotational inertia, angular displacement, angular velocity, angular acceleration and angular momentum.

For classical electromagnetism, Maxwell's equations are involved in the kinematics. The dynamics of classical systems where both mechanics and electromagnetism are involved include the combination of Newton's laws, Maxwell's equations and the Lorentz force.

History

As early as Galileo's time, scientists attempted to demonstrate Earth's rotation by dropping objects and measuring how far eastward they landed. However, these efforts were so crude and inaccurate that it became very difficult to be conclusive.

Timelines

1742: *Colin Maclaurin* discovered uniformly rotating self-gravitating spheroids.

1760: *Euler* 3-body problem.

1834: *Carl Jacobi* discovered his uniformly rotating self-gravitating ellipsoids.

1834: *Louis Poinsot* established an instance of the intermediate axis theorem.

Applications (Real Life / Industrial)

Around a fixed axis, the simplest case of rotation is that of constant angular speed when the total torque becomes zero. For example, the Earth rotating around its axis and there is very little friction. For a fan, the motor applies a torque in order to compensate for friction. Similar to the fan, equipments found in the mass production manufacturing industry demonstrate rotation effectively around a fixed axis. A multi-spindle lathe is used for rotating the material on its axis for the purpose of effectively increasing the production of cutting, deformation and turning.

Case Study (Environmental / Sustainability / Social / Ethical Issues)

Let us consider tornadoes and their devastating nature. It has a comparable speed of the cyclone that engulfs the surroundings but besides the speed there is something more. A tornado is a mixture of force, power, and energy which govern the rotational motion of a tornado at the time when causes the destructions. In rotational motion, the particles of the object while moving follow a circular path. Every particle in the rigid body moves in a circular path along a plane perpendicular to the axis and has its centre on the same axis. There can be two instances of rotational motion one about the fixed axis and the other about an unfixed axis. The popular example of rotation around a fixed axis is the fan.

Inquisitiveness and Curiosity Topics

'Particle' represents a point mass at some position in space. It can move about but it has no characteristic orientation or rotational inertia. It is characterized only by its mass. With a view to calculating the orbit of a satellite we have no need to know the orientation of the satellite and as we know that the satellite is very small compared with the dimensions of its orbit.

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6 Dynamics of a Rigid Body

UNIT SPECIFICS

In this unit we have considered the following aspects:

- Three-dimensional rigid body motion in terms of
    - Angular velocity vector,
    - Its rate of change,
    - Moment of inertia tensor;
- Three-dimensional motion of a rigid body wherein
    - All points move in a coplanar manner,
- Calculation of moment of inertia of symmetric bodies;
- Ellipsoid of inertia;
- Euler's equation of motion.

The practical applications of the topics are discussed for generating further curiosity and creativity as well as improving problem solving capacity.

Besides giving a large number of multiple choice questions as well as questions of short and long answer types marked in two categories following lower and higher order of Bloom's taxonomy, assignments through a number of numerical problems, a list of references and suggested readings are given in the unit so that one can go through them for practice. It is important to note that for getting more information on various topics of interest some QR codes have been provided in different sections which can be scanned for relevant supportive knowledge.

After the related practical, based on the content, there is a "Know More" section. This section has been carefully designed so that the supplementary information provided in this part becomes beneficial for the users of the book. This section mainly highlights the initial activity, examples of some interesting facts, analogy, history of the development of the subject focusing the salient observations and finding, timelines starting from the development of the concerned topics up to the recent time, applications of the subject matter for our day-to-day real life or/and industrial applications on variety of aspects, case study related to environmental, sustainability, social and ethical issues whichever applicable, and finally inquisitiveness and curiosity topics of the unit.

RATIONALE

This unit on Dynamics of a Rigid Body will help students to get a theoretical idea about the three-dimensional rigid body motion and its rate of change and moment of inertia tensor. It also covers three-dimensional motion of a rigid body wherein all points move in a coplanar manner. The topics will develop a clear concept of the dynamics of rigid body which is considered as one of the very important aspects in mechanics.

Dynamics of rigid body is the study of the motion in space of one or many bodies wherein deformation is ignored. On a rigid body all lines have same angular velocity and same angular acceleration. Rigid motions are decomposed into the translation of an arbitrary point, followed by a rotation about that point. Thus there are two types of motion in a rigid body, viz., translational motion and rotational motion. Dimensions of the body are unimportant to the description of its motion, *e.g.*, in planetary motion, the planets are considered as particles. Rigid body has a finite size but it does not deform.

PRE-REQUISITES

Mathematics: Calculus (Class XII)

Physics: Rotational Motion (Class XII)

UNIT OUTCOMES

List of outcomes of this unit is as follows:

U6-O1: Explain the role of degrees of freedom to specify a rigid body completely

U6-O2: Describe rolling of an object with the help of translational as well as rotational concepts

U6-O3: Calculate principal moments of inertia and products of inertia of some symmetric bodies

U6-O4: Apply the concept of ellipsoid of inertia and tensor of inertia in complex rigid body problem

U6-O5: Analyze rigid body problems using Euler's equations of motion

Unit-6 Outcomes	EXPECTED MAPPING WITH COURSE OUTCOMES (*1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation*)					
	CO-1	CO-2	CO-3	CO-4	CO-5	CO-6
U6-O1	3	-	1	-	-	-
U6-O2	3	2	1	-	-	-
U6-O3	3	1	1	-	-	-
U6-O4	3	2	1	-	-	-
U6-O5	3	2	1	-	-	-

6.1 INTRODUCTION

The bodies are called rigid, in general, when they do not make any deformation under the action of applied forces. A rigid body can be defined as one in which the distance between any two of its constituent particles always remains constant. A perfect rigid body is an idealization since all bodies are made up of atoms and molecules which are in a state of incessant motion. Further, elastic deformations may also occur in bodies changing their shapes and sizes. As all such microscopic displacements are negligibly small so any given body may be treated as rigid. A rigid body can have two types of motions viz. translational and rotational. The motion of a rigid body can be completely specified if its position and orientation are known.

For such specification, what we require are any three non collinear points in the body. If however the body is fixed at one point, it cannot have any translational motion, but it can rotate about an axis passing through that fixed point. If another point of the body is also fixed, the body is capable of rotation about the axis through the two fixed points. In that case, the coordinates of the third point, not on the axis, would locate the body completely in space. Usually, any point in space is specified in terms of its 3 coordinates in some chosen reference frame. Therefore, for the complete specification of a rigid body 9 (3×3) coordinates are essential. But they are not all independent but being associated to the equations of constrains:

$$\left(\vec{r}_i - \vec{r}_j\right)^2 = \text{constant} \tag{6.1}$$

where $\vec{r}_i$ or $\vec{r}_j\,(i, j = 1,2,3...)$ represents the position vector of the three points in the rigid body. So there are only 6 independent coordinates. Or in other words, a rigid body is said to have 6 *degree of freedom*.

6.2 ROTATION OF A RIGID BODY

Rigid body is a finite object where distances between the constituent particles are found to be constant. If we consider a rigid body in pure rotational motion, body particles will rotate through a particular angle in course of identical time interval, thus making all particles of same angular velocity and angular acceleration.

We consider a rigid body which is rotating through a particular axis having a constant angular velocity $\vec{\omega}$. Its direction is taken as parallel to the axis of rotation with sign as directed by right-handed screw rule.

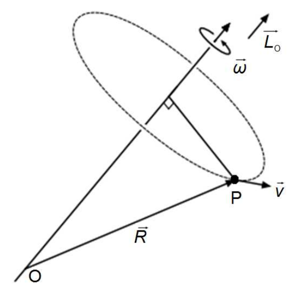

Fig. 6.1: Rigid body rotation

Any non-axis point particle P with radius vector $\vec{R}$ then describes a circle with its plane normal to the axis. Its linear velocity $\vec{v}$ is at right angles to both $\vec{R}$ and $\vec{\omega}$.

$$\therefore \qquad \vec{v} = \vec{\omega}\times\vec{R} \tag{6.2}$$

If the particle is of mass m, then its momentum is given by

$$\vec{p} = m\vec{v}$$

$$= m(\vec{\omega}\times\vec{R}) \tag{6.3}$$

Angular momentum of the particle about O [Fig. 6.1] is

$$\overrightarrow{L_0} = \vec{R}\times\vec{p} = \vec{R}\times m(\vec{\omega}\times\vec{R}) = m\vec{R}\times(\vec{\omega}\times\vec{R}) = m\left[\left(\vec{R}.\vec{R}\right)\vec{\omega} - \left(\vec{R}.\vec{\omega}\right)\vec{R}\right]$$

$$= m\left[R^2\vec{\omega} - \left(\vec{R}.\vec{\omega}\right)\vec{R}\right] \tag{6.4}$$

If the origin be shifted to some other point, then the angular momentum $\vec{L}$ about that point is obtained by putting $R = r$, the distance of the particle from the point under consideration. Taking the angle between $\vec{R}$ and $\vec{\omega}$ as $\frac{\pi}{2}$ we have:

Angular momentum vector, $\vec{L} = mr^2\vec{\omega}$

and total angular momentum $= \sum mr^2\vec{\omega} = I\vec{\omega}$ (6.5)

where $I = \sum mr^2$, the summation extending over all the particles comprising the body.

Kinetic energy of the rotating body

$$T = \frac{1}{2}\sum m\left|\vec{v}\right|^2 = \frac{1}{2}\sum m\left|\vec{\omega}\times\vec{R}\right|^2 = \frac{1}{2}\sum m\omega^2R^2\sin^2\theta = \frac{1}{2}\sum mr^2\omega^2 = \frac{1}{2}I\omega^2 \tag{6.6}$$

In linear motion, the momentum and kinetic energy of a body of mass M are respectively MV and $\frac{1}{2}MV^2$ and their analogues in rotation, the angular momentum and rotational kinetic energy, are given by $I\omega$ and $\frac{1}{2}I\omega^2$.

The quantity $I = \sum m_i r_i^2$ is called moment of inertia (MI) of the body with reference to the rotational axis.

6.3 ROLLING WHEEL

If a wheel is designed to roll as part of a mechanism, then it is called rolling wheel. Let, a wheel moves on rolling [Fig. 6.2]. So, there will be two types of motion associated with the rolling wheel. These are: i) translational motion and ii) rotational motion. So, due to this there will be two types of kinetic energies associated with the body, viz., the translational kinetic energy $T_{tran} = \frac{1}{2}mv^2$ and the rotational kinetic energy $T_{rot} = \frac{1}{2}I\omega^2$.

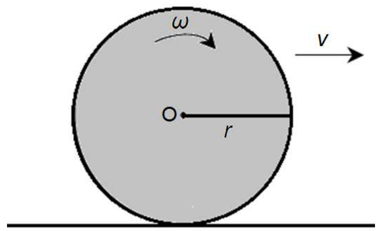

Fig. 6.2: Motion of a rolling wheel

Thus the total kinetic energy will be,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}mK^2\frac{v^2}{r^2} = \frac{mv^2}{2}\left(1+\frac{K^2}{r^2}\right) \tag{6.7}$$

Here, K is the radius of gyration, $I = mK^2$ and $\omega = \frac{v}{r}$.

SCAN ME

for more on the motion of a wheel

Example 6.1 An annular disc of mass 150 g and radii 20 cm and 25 cm rolls such that its centre has a velocity of 50 cm /s. Calculate its kinetic energy.

Solution

The total kinetic energy is the sum of the translational kinetic energy and rotational kinetic energy. Hence, it is given by,

$$\text{KE} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{4}m\left(r_1^2 + r_2^2\right)\frac{v^2}{r_2^2}$$

or, $$\text{KE} = \frac{1}{2}\times 150\times 50^2 + \frac{1}{4}\times 150\times\left(25^2 + 20^2\right)\times\frac{50^2}{25^2} = 34.12\times 10^4\,\text{erg.}$$

EXAMPLE 6.1

6.4 BODY ROLLING DOWN

If a body is kept on an inclined plane, it attempts to slip down and so a static friction acts upwards which provides a torque that causes the body to rotate.

Let a body of radius r roll down, without slip, along an inclined plane of inclination θ [Fig. 6.3]. Suppose it starts rolling from rest at A and reaches B after rolling over a distance l along the inclination, acquiring a translational velocity v. Now, the KE acquired by the body at B is the loss in its PE when falling from A to B, as there is no slipping there is no dissipation of energy. The KE at B has two parts, viz., translational and rotational and, if ω be the angular velocity at B, it is given by

$$T_{tran} + T_{rot} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}mK^2\omega^2 \tag{6.8}$$

where K is the radius of gyration of the body about the axis.

$$\text{Kinetic energy, KE} = \frac{1}{2}mv^2\left(1 + \frac{K^2}{r^2}\right) \quad (\because v = r\omega) \tag{6.9}$$

The loss of potential energy in rolling from $\text{A} \to \text{B}$ is

$$mgh = mgl\sin\theta\,.$$

$$\therefore \quad \frac{1}{2}mv^2\left(1 + \frac{K^2}{r^2}\right) = mgl\sin\theta$$

or, $$v^2\left(1 + \frac{K^2}{r^2}\right) = 2gl\sin\theta$$

$$\therefore \quad v^2 = 2l\frac{g\sin\theta}{\left(1 + \frac{K^2}{r^2}\right)} \tag{6.10}$$

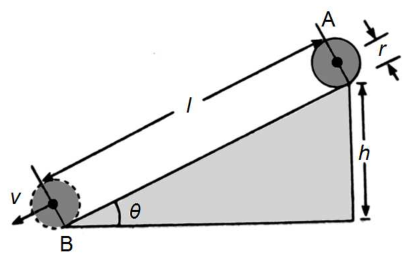

Fig. 6.3: Body rolling down an inclined plane

which at once shows that the acceleration of the body rolling down is

$$f = \frac{g\sin\theta}{\left(1+\dfrac{K^2}{r^2}\right)} \tag{6.11}$$

The value of the radius of gyration, K is different for different bodies. The following are some of the important cases:

i) A solid sphere: $K^2 = \frac{2}{5}r^2,$

$$\therefore \qquad f = \frac{5}{7}g\sin\theta.$$

ii) A solid cylinder: $K^2 = \frac{1}{2}r^2,$

$$\therefore \qquad f = \frac{2}{3}g\sin\theta.$$

iii) A hollow cylinder: $K^2 = r^2,$

$$\therefore \qquad f = \frac{1}{2}g\sin\theta.$$

SCAN ME

for more on rolling and sliding down an incline

EXAMPLE 6.2

Example 6.2 A fly-wheel of mass 100 kg and diameter 1 m makes 210 rev/m. Assuming the mass to be concentrated at the rim, find the moment of inertia and energy of the fly-wheel.

Solution Moment of inertia of the fly-wheel is

$$\mathrm{I} = \mathrm{MR}^2 = 100 \times \left(\frac{1}{2}\right)^2 = 25 \text{ kg-m}^2.$$

and energy of the fly-wheel is

$$\mathrm{E} = \frac{1}{2}\mathrm{I}\omega^2 = \frac{1}{2}\times 25 \times \left(\frac{2\pi\times 210}{60}\right)^2 \text{ J} = 60.4\times 10^2 \text{ J}.$$

EXAMPLE 6.3

Example 6.3 A fly-wheel of mass 100 kg and diameter 1 m is in the shape of a circular disc. Find the amount of work done if its speed of revolution is increased from 10 rps to 20 rps.

Solution

Here, $\quad \omega_1 = 2\pi\times 10 = 20\pi \text{rad/s}$ and $\omega_2 = 2\pi \times 20 = 40\pi \text{rad/s}.$

and $\quad \mathrm{I} = \mathrm{MR}^2 = 100\times\left(\frac{1}{2}\right)^2 = 25 \text{ kg-m}^2.$

So, the amount of work done if its speed of revolution is increased from 10 rps to 20 rps is $\quad \mathrm{W} = \mathrm{E}_2 - \mathrm{E}_1 = \frac{1}{2}\mathrm{I}\omega_2^2 - \frac{1}{2}\mathrm{I}\omega_1^2 = \frac{25}{2}\left(160\pi^2 - 40\pi^2\right) = 14.8\times 10^3 \text{ J}.$

6.5 ELLIPSOID OF INERTIA

The MI of a body obviously varies with the position and the direction of the axis in relation to the body. Parallel axis theorem only discusses the variation of MI for different positions of the axis; its direction however remains unaltered. We shall now discuss the most general variation in MI for different directions of the axis.

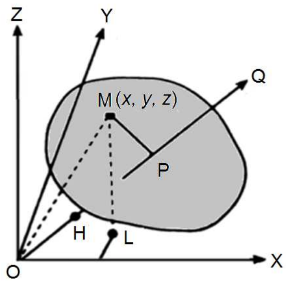

Fig. 6.4: Ellipsoid of inertia

Let a system of rectangular Cartesian axes OXYZ be fixed with respect to the body and OQ be any axis through the origin O [Fig. 6.4] and having the direction determined by the direction cosines *l, m* and *n*. We shall evaluate the MI about OQ as axis. Let M (*x, y, z*) be any point on the body and a particle of mass m' of the body be located there.

From Fig. 6.4, the MI of m' about OQ is given by

$$dI = m' \times \mathrm{MP}^2 = m'\left(\mathrm{OM}^2 - \mathrm{OP}^2\right)$$

But, $\mathrm{OM}^2 = x^2 + y^2 + z^2$

and we have OP^2 = square of projection of OM on $\mathrm{OQ} = (lx + my + nz)^2$

So, the MI of m' about OQ is

$$\begin{aligned} dI &= m'\left[\left(x^2 + y^2 + z^2\right) - (lx + my + nz)^2\right] \\ &= m'\left[x^2\left(1 - l^2\right) + y^2\left(1 - m^2\right) + z^2\left(1 - n^2\right) - 2lmxy - 2mnyz - 2nlzx\right] \\ &= m'\left[x^2\left(m^2 + n^2\right) + y^2\left(n^2 + l^2\right) + z^2\left(l^2 + m^2\right) - 2lmxy - 2mnyz - 2nlzx\right] \end{aligned}$$

using the well known direction cosine relation: $l^2 + m^2 + n^2 = 1$.

So, the MI of the entire body about OQ is

$$\begin{aligned} I = \sum dI &= \sum m'\{x^2\left(m^2 + n^2\right) + y^2\left(n^2 + l^2\right) + z^2\left(l^2 + m^2\right) - 2lmxy - 2mnyz - 2nlzx\} \\ &= l^2 \sum m'\left(y^2 + z^2\right) + m^2 \sum m'\left(z^2 + x^2\right) + n^2 \sum m'\left(x^2 + y^2\right) \\ &\quad - 2lm \sum m'xy - 2mn \sum m'yz - 2nl \sum m'zx \end{aligned}$$

or, $$I = I_{xx}l^2 + I_{yy}m^2 + I_{zz}n^2 + 2I_{xy}lm + 2I_{yz}mn + 2I_{zx}nl \tag{6.12}$$

where $I_{xx} = \sum m'\left(y^2 + z^2\right) =$ MI of the body about X-axis,

$I_{yy} = \sum m'\left(z^2 + x^2\right) =$ MI of the body about Y-axis and

$I_{zz} = \sum m'\left(x^2 + y^2\right) =$ MI of the body about Z-axis.

Again, $I_{xy} = -\sum m'xy, I_{yz} = -\sum m'yz, I_{zx} = -\sum m'zx$ and they are called the products of inertia with respect to axes X and Y, Y and Z, Z and X or simply about XY, YZ and ZX axis respectively.

EXAMPLE 6.4

Example 6.4 The 3 principal moment of inertia of a body at a point are given as 200, 300 and 450 g-cm^2. Write down the equation of the ellipsoid of inertia at that point.

Solution

The equation of ellipsoid of inertia at a point in reference to the principal axes is given by

$$I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 = \text{constant}$$

By the problem, we have here

$$I_{xx} = 200, I_{yy} = 300, I_{zz} = 450$$

So, the above equation can written as $200x^2 + 300y^2 + 450z^2 = \text{constant}$

or, $$4x^2 + 6y^2 + 9z^2 = \text{constant}$$

which is the required equation of the ellipsoid of inertia.

6.6 MOMENTS AND PRODUCTS OF INERTIA

6.6.1 Thin Uniform Rod

For a thin uniform straight rod along Y-axis (Fig. 6.5), we have $x_i = z_i = 0$. Let O be the origin, represents the centre of mass (CM) of the rod:

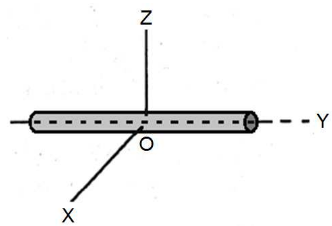

Fig. 6.5: MI calculation for thin uniform rod

So, $$I_{xx} = I_{zz} = \sum m_i y_i^2 \quad , \; I_{yy} = 0 \tag{6.13}$$

For continuous mass distribution,

$$I_{xx} = I_{zz} = \int y^2 dm = \int_{-l/2}^{l/2} y^2 \lambda dy$$

$$= \lambda \int_{-l/2}^{l/2} y^2 dy = \lambda \frac{l^3}{12} = \frac{Ml^2}{12} \tag{6.14}$$

$\lambda l = M$ = mass per unit length and $\lambda l = M$ = the mass of the rod.

6.6.2 Rectangular Lamina

Let l be the length, b be the breadth and O the CM (Fig. 6.6) of the rectangular lamina. The lamina being very thin, $z = 0$.

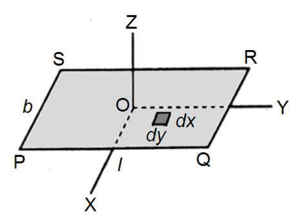

Fig. 6.6: MI calculation for a rectangular lamina

$$I_{xx} = \int (y^2 + z^2) dm = \iint y^2 dx dy . \sigma$$

$$= \sigma \int_{-l/2}^{l/2} y^2 dy \int_{-b/2}^{b/2} dx = \frac{Ml^2}{12} \tag{6.15}$$

Here σ = surface density, $bl\sigma = M$ = mass of the lamina.

6.6.3 Circular Lamina

We take both X-axis and Y-axis in the plane of the lamina and the Z-axis is taken perpendicular through its centre of mass O (Fig. 6.7).

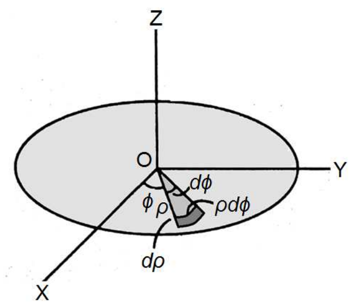

Fig. 6.7: MI calculation for a circular lamina

Using cylindrical coordinates:

$$x = \rho\cos\phi, = \rho\sin\phi, z = z\ ,$$

elemental area, $dS = d\rho.\rho d\phi$,

elemental volume, $dV = d\rho.\rho d\phi dz$

we get, $$I_{zz} = \int (x^2 + y^2)dm = \int \rho^2 d\rho\rho d\phi\sigma$$

$$= \sigma\int_0^R \rho^3 d\rho \int_0^{2\pi} d\phi = \sigma\frac{R^4}{4}2\pi$$

$$= \frac{MR^2}{2} \tag{6.16}$$

and $$I_{xx} = \int (y^2 + z^2)dm = \int y^2 dm = \int\int \rho^2 \sin^2\phi d\rho\rho d\phi\sigma$$

$$= \sigma\int_0^R \rho^3 d\rho \int_0^{2\pi} \sin^2\phi d\phi = \sigma\frac{R^4}{4}\pi = \frac{MR^2}{4} \tag{6.17}$$

Similarly, $$I_{yy} = \frac{MR^2}{4} \tag{6.18}$$

6.6.4 Cylinder

We take Z-axis as the axis of the cylinder, O being the CM (Fig. 6.8). We shall also use the cylindrical coordinates. So,

$$I_{zz} = \int \rho_0(x^2 + y^2)dV = \rho_0\int \rho^2 d\rho\rho d\phi dz$$

$$= \rho_0\int_0^R \rho^3 d\rho \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} dz$$

$$= \rho_0\frac{R^4}{4}2\pi h = \frac{MR^2}{2} \tag{6.19}$$

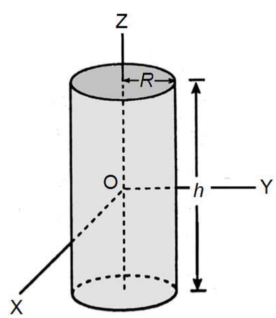

Fig. 6.8: MI calculation for a cylinder

where M = mass of cylinder, ρ_0 = density.

and $$I_{xx} = \int (y^2 + z^2)\rho_0 d\rho\rho d\phi dz$$

$$= \int\left(\rho^2\sin^2\phi + z^2\right)\rho_0 d\rho\rho d\phi dz$$

$$= \rho_0 \left[\int_0^R \rho^3 d\rho \int_0^{2\pi} \sin^2 \phi d\phi \int_{-h/2}^{h/2} dz + \int_0^R \rho d\rho \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} z^2 dz \right]$$

$$= \rho_0 \left[\frac{R^4}{4} \pi h + \frac{R^2}{2} 2\pi \frac{h^2}{12} \right]$$

$$= M \left[\frac{R^4}{4} + \frac{h^2}{12} \right] \tag{6.20}$$

From symmetry, $I_{yy} = I_{xx}$.

$$I_{xy} = -\int xydm = -\iiint \rho \cos\phi \rho \sin\phi d\rho . \rho d\phi . dz \rho_0$$

$$= -\rho_0 \int_0^R \rho^3 d\rho \int_0^{2\pi} \sin\phi \cos\phi d\phi \int_{-h/2}^{h/2} dz = 0 \quad (\because \phi \text{ integration} = 0) \tag{6.21}$$

Similarly, $I_{xz} = I_{yz} = 0$.

6.7 ANGULAR MOMENTUM OF RIGID BODY

The angular momentum of a rotating rigid body is directed along the rotational axis. The torque directed along the rotational axis is given by the time rate of change of the angular momentum.

Let a rigid body consisting of n-particles having masses $m_i (i = 1, 2, 3, ..., n)$ rotate with an instantaneous angular velocity $\vec{\omega}$. Suppose one of the points is fixed, implying that there is no translational motion. Let us now compute the angular momentum of the body and its kinetic energy due to rotation.

So, for the i^{th} particle the linear velocity $\vec{v_i}$ of mass m_i is $\vec{v_i} = \vec{\omega} \times \vec{r_i}$.

Total angular momentum $\vec{L}$ is obtained by summing up all the angular momentum of individual particles as,

$$\vec{L} = \sum_i \vec{r_i} \times m_i \vec{v_i} = \sum_i m_i \vec{r_i} \times \left(\vec{\omega} \times \vec{r_i} \right) = \sum_i m_i r_i^2 \omega - \sum_i m_i \left(\vec{r_i} . \omega \right) \vec{r_i} \tag{6.22}$$

The above equation may be written in component forms and the x component of angular momentum is given by

$$L_x = \sum_i m_i (r_i^2 - x_i^2) \omega_x - \sum_i m_i r_i y_i \omega_y - \sum_i m_i r_i x_i \omega_x \tag{6.23}$$

Introducing the following symbols for the coefficients of $\omega_x, \omega_y, \omega_z$ we can write

$$I_{xx} = \sum_i m_i (r_i^2 - x_i^2) = \sum_i m_i (y_i^2 + x_i^2);\ I_{yy} = \sum_i m_i (z_i^2 + x_i^2);\ I_{zz} = \sum_i m_i (x_i^2 + y_i^2)$$

and $$I_{xy} = -\sum_i m_i x_i y_i = I_{yx};\ I_{yz} = -\sum_i m_i y_i z_i = I_{zy};\ I_{zx} = -\sum_i m_i z_i x_i = I_{xz}$$

Similarly, we obtain

$$L_{xx} = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \tag{6.24 (a)}$$

$$L_{xy} = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \tag{6.24 (b)}$$

$$L_{xz} = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \tag{6.24 (c)}$$

The quantities I_{xx}, I_{yy} and I_{zz} are called the moments of inertia about X, Y and Z axes respectively. I_{xy}, I_{yz} and I_{zx} are called the products of inertia.

Generalizing the above set of Eqs. (6.24) we get

$$L_i = \sum_i I_{ij}\omega_j \quad (i = j = 1, 2, 3) \tag{6.25}$$

The coefficients I_{ij} can be calculated from the distribution of the particles about the different axes.

6.8 KINETIC ENERGY OF RIGID BODY

The kinetic energy of an object represents the energy that it acquires due to its motion. In classical mechanics, if we consider a non-rotating object, the kinetic energy can be obtained by considering the mass of the object and its speed by a well known equation $E = \frac{1}{2}mv^2$ while in relativistic mechanics it may be considered as a good approximation when v is too small compared to the speed of light.

The kinetic energy of the rigid body is given by $T = \sum_i \frac{1}{2}m_i v_i^2$

or, $$2T = \sum_i m_i \left|\vec{v}_i\right|^2 = \sum_i m_i \left(\vec{\omega}\times\vec{r}_i\right).\left(\vec{\omega}\times\vec{r}_i\right) = \sum_i m_i \vec{\omega}\left[\vec{r}_i \times\left(\vec{\omega}\times\vec{r}_i\right)\right]$$

$$T = \vec{\omega}.\sum_i m_i \vec{r}_i \times\left(\vec{\omega}\times\vec{r}_i\right) = \vec{\omega}.\sum_i m_i \vec{r}_i \times \vec{v}_i$$

$\therefore$ $$2T = \vec{\omega}.\vec{L}$$

$\therefore$ Kinetic energy: $$T = \frac{1}{2}\vec{\omega}.\vec{L} = \frac{1}{2}\sum_i \omega_i L_i = \frac{1}{2}\sum_{ij} I_{ij}\omega_i\omega_j \tag{6.26 (a)}$$

$$= \frac{1}{2}\left[I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2I_{xy}\omega_x\omega_y + 2I_{yz}\omega_y\omega_z + 2I_{zx}\omega_z\omega_z\right] \tag{6.26 (b)}$$

Special case: Let the body rotate about the Z-axis with angular velocity ω.

$\therefore$ $$\omega_z = \omega,\ \omega_x = \omega_y = 0$$

So, the kinetic energy becomes

$$T = \frac{1}{2}I_{zz}\omega_z^2 = \frac{1}{2}I\omega^2 \tag{6.27}$$

where I is the MI of the body about the Z-axis.

The components of angular momenta, are thus given by

$$L_x = I_{xz}\omega_z, L_y = I_{yz}\omega_z, L_z = I_{zz}\omega_z \tag{6.28}$$

EXAMPLE 6.5

Example 6.5 Show that $T = \frac{1}{2}\vec{L}.\vec{\omega}$

Solution

Now, $\frac{1}{2}\vec{L}.\vec{\omega} = \frac{1}{2}\left(\hat{i}L_x + \hat{j}L_y + \hat{k}L_z\right)\cdot\left(\hat{i}\omega_x + \hat{j}\omega_y + \hat{k}\omega_z\right)$

or, $\frac{1}{2}\vec{L}.\vec{\omega} = \frac{1}{2}\left(\hat{i}I_{xx}\omega_x + \hat{j}I_{yy}\omega_y + \hat{k}I_{zz}\omega_z\right)\cdot\left(\hat{i}\omega_x + \hat{j}\omega_y + \hat{k}\omega_z\right)$

$$= \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right) = T$$

$\therefore$ $T = \frac{1}{2}\vec{L}.\vec{\omega}.$

6.9 INERTIA TENSOR

The nine terms I_{ij} *i.e.*, $I_{xx}, I_{xy}, I_{xz}, I_{yx}, I_{yy}, I_{yz}, I_{zx}, I_{zy}, I_{zz}$ may be looked upon as the components of the MI of the rigid body. Note that each of the components is a scalar quantity and has the dimension [ML^2]. Since there are in all 9 or 3×3 components, the physical quantity, MI, is a tensor of rank 2 and is known as the inertia tensor or MI tensor symbolized by $\overleftrightarrow{I}$. When written in matrix notation we get,

$$\overleftrightarrow{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \tag{6.29}$$

The inertia tensor is symmetric, *i.e.*, $I_{ij} = I_{ji}$. So, the numbers of independent components are only six. The equation $L_i = \sum_i I_{ij}\omega_j$ representing the angular momentum, may now be written as the dot product of inertia tensor and the angular velocity vector, *i.e.*,

$$\vec{L} = \overleftrightarrow{I}.\vec{\omega} \tag{6.30}$$

and the equation for kinetic energy $T = \frac{1}{2}\vec{\omega}.\vec{L}$ take the form

$$T = \frac{1}{2}\vec{\omega}.\overleftrightarrow{I}.\vec{\omega} \tag{6.31}$$

6.10 EULER'S EQUATION OF MOTION

Euler's laws of motion, in classical mechanics, are equations extending laws of motion given by Newton as associated with point particles for the motion of a rigid body. All these laws were developed by Leonhard Euler after about 50 years of Newton laws.

Let $\vec{N}$ be the torque acting on a rigid body, one point of which is fixed. Then the equation of its rotational motion is a fixed in space or for an inertial frame we can write,

$$\left(\frac{d\vec{L}}{dt}\right)_f = \vec{N}$$

But $\frac{d}{dt}$ operator in a fixed frame is related to that in a rotating frame by

$$\left(\frac{d}{dt}\right)_f = \left(\frac{d}{dt}\right)_r + \vec{\omega}\times \tag{6.32}$$

operating upon vectors.

The rigid body rotates with an angular velocity $\vec{\omega}$ and fixing a frame of reference in it. We can write for the body frame of reference as

$$\vec{N} = \left(\frac{d\vec{L}}{dt}\right)_f = \left(\frac{d}{dt}\right)_r + \vec{\omega}\times\vec{L} = \overleftrightarrow{I}\cdot\frac{d\vec{\omega}}{dt} + \vec{\omega}\times\vec{L} = \overleftrightarrow{I}\cdot\dot{\vec{\omega}} + \vec{\omega}\times\vec{L} \tag{6.33}$$

where, $\overleftrightarrow{I}$ is a constant relative to the body axes. If ω_1, ω_2 and ω_3 be the components of $\vec{\omega}$ along the principal axes and $\hat{a}_1, \hat{a}_2, \hat{a}_3$ be the unit vectors along them, we have

$$\vec{L} = I_1\omega_1\hat{a}_1 + I_2\omega_2\hat{a}_2 + I_3\omega_3\hat{a}_3 \tag{6.34}$$

and

$$\overleftrightarrow{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{6.35}$$

The axes of the body frame are coinciding with the principal axes of the body. So, in component form we can write;

$$N_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 \tag{6.36 (a)}$$

$$N_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 \tag{6.36 (b)}$$

$$N_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 \tag{6.36 (c)}$$

The above three simultaneous differential equations are known as the Euler's dynamical equations of a rigid body.

UNIT SUMMARY

- ***Rotation of a rigid body***

$$T=\frac{1}{2}\sum m\left|\vec{v}\right|^2=\frac{1}{2}\sum m\left|\vec{\omega}\times\vec{R}\right|^2=\frac{1}{2}\sum m\omega^2R^2\sin^2\theta=\frac{1}{2}\sum mr^2\omega^2=\frac{1}{2}I\omega^2$$

- ***Rolling wheel***

$$E=\frac{1}{2}mv^2+\frac{1}{2}I\omega^2=\frac{1}{2}mv^2+\frac{1}{2}mK^2\frac{v^2}{r^2}=\frac{mv^2}{2}\left(1+\frac{K^2}{r^2}\right)$$

- ***Body rolling down an inclined plane without slip***

$$v^2=2l\frac{g\sin\theta}{\left(1+\frac{K^2}{r^2}\right)}$$

$$f=\frac{g\sin\theta}{\left(1+\frac{K^2}{r^2}\right)}$$

- ***Ellipsoid of inertia***

$$I=I_{xx}l^2+I_{yy}m^2+I_{zz}n^2+2I_{xy}lm+2I_{yz}mn+2I_{zx}nl$$

- ***Kinetic energy of rigid body***

$$T=\frac{1}{2}\vec{\omega}.\vec{L}=\frac{1}{2}\sum_i\omega_iL_i=\frac{1}{2}\sum_{ij}I_{ij}\omega_i\omega_j$$

- ***Angular momentum of rigid body***

$$L_i=\sum_i I_{ij}\omega_j \quad (i=j=1,2,3)$$

- ***Inertia tensor***

$$\overleftrightarrow{I}=\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

- ***Euler's equation of motion***

$$N_1=I_1\dot{\omega}_1-\left(I_2-I_3\right)\omega_2\omega_3$$

$$N_2=I_2\dot{\omega}_2-\left(I_3-I_1\right)\omega_3\omega_1$$

$$N_3=I_3\dot{\omega}_3-\left(I_1-I_2\right)\omega_1\omega_2$$

EXERCISES

Multiple Choice Questions

6.1 A sphere cannot roll on a
(a) smooth horizontal surface (b) smooth incline surface
(c) rough horizontal surface (d) rough incline surface

6.2 A disc, a hollow sphere and a solid sphere are placed and released from the top of a small incline. If all the three bodies have the same radius and mass, then most time to reach the bottom will be taken by
(a) solid sphere (b) disc (c) hollow sphere (d) all reach at a time

6.3 A solid sphere rolls down from two different inclined planes of the same heights but different inclinations. In each case, the ball will reach the bottom
(a) with the same speed (b) with different speed
(c) with different speed but same time (d) immediately

6.4 A rigid body has how many degrees of freedom?
(a) one (b) two (c) three (d) six

6.5 A hook of radius 2 m weighs 100 kg rolls along a horizontal floor so that its centre if mass has a speed of 20 cm/s. How much work has to be done to stop it?
(a) 2 J (b) 6 J (c) 5 J (d) 4 J

6.6 A smooth sphere A is moving on a frictionless horizontal plane with angular speed ω and the centre of mass velocity is u. It collides elastically and head on with an identical sphere B at rest. Neglect friction everywhere. After the collision, their angular speeds are ω_A and ω_B respectively. Then
(a) $\omega_A < \omega_B$ (b) $\omega_A = \omega_B$ (c) $\omega_A = \omega$ (d) $\omega_B = \omega$

6.7 One solid sphere A and another hollow sphere B are of same mass and same outer radii. Their moments of inertia about their diameters are respectively I_A and I_B, such that
(a) $I_A = I_B$ (b) $I_A > I_B$ (c) $I_A < I_B$ (d) $I_A/I_B = \rho_A/\rho_B$

6.8 For the rotation of a rigid body the directions of the angular velocity and the angular momentum are
(a) same (b) different (c) perpendicular (d) parallel

6.9 The rank of moment of inertia tensor is
(a) one (b) two (c) three (d) zero

Answers of Multiple Choice Questions

6.1 (c), 6.2 (d), 6.3 (a), 6.4 (d), 6.5 (d), 6.6 (c), 6.7 (c), 6.8 (c), 6.9 (b)

Short and Long Answer Type Questions

Category I

6.1 Explain the principle of moment.
6.2 Compare parallel and perpendicular axis theorem and prove them.

6.3 Clarify the following:
i) ellipsoid of inertia and ii) tensor of inertia.

6.4 Give comparative statements between i) parallel and ii) perpendicular axes theorems.

6.5 A rigid body rotates about a fixed axis with angular velocity ω. Calculate the kinetic energy of rotation and define moment of inertia.

6.6 Find the moment of inertia and ellipsoid of inertia of a solid sphere of radius r with centre at the origin O.

6.7 Derive an expression for the linear acceleration of a sphere rolling down an incline, without slip and starting from rest.

6.8 A rigid body rotates with an angular velocity ω about an axis through the origin O and having direction cosines (l, m, n). Show that the moment of inertia of the rigid body about the axis is

$$I_{lmn} = I_{xx}l^2 + I_{yy}m^2 + I_{zz}n^2 + 2I_{xy}lm + 2I_{yz}mn + 2I_{zx}nl.$$

6.9 Establish that the expression for the kinetic energy of rotation for a rigid body is given by $T = \frac{1}{2}\vec{\omega}.\vec{L}$, where the symbols are of their usual significance.

Category II

6.10 A rigid body is rotating with angular velocity vector $\vec{\omega}$ about an axis through the origin O having direction cosines (l, m, n).

(a) Find an expression for the kinetic energy of rotation. How the expression is modified for principal axes system?

(b) Find an expression for angular momentum of the rigid body about the axis (l, m, n). How the expression should be modified for principal axes system?

6.11 If the angular momentum of a body is zero about a point, is it necessary that it will also be zero at any other point? Justify your answer.

6.12 Can an object be in pure translation as well as in pure rotation? Justify your answer.

Numerical Problems

6.1 A solid sphere of mass 100 g and radius 2.5 cm rolls, without sliding, with a uniform velocity of 10 cm/s along a straight line on a horizontal table. Calculate its total energy. [*Ans:* 7000 erg]

PRACTICAL

To qualitatively verify typical motions of a Gyroscope

Aim

i) To qualitatively observe some of the motions of a gyroscope
ii) To verify a simple motion quantitatively

Theory

A gyroscope is a rigid rotating symmetric object about one axis. The gyroscopic motion can be interpreted by applying a force $\vec{F} = m\vec{a}$ to all the particles with which the rigid body made. Majority of the forces acts between the particles to make it rigid. The overall motion is expressed as,

$$\vec{\tau} = \frac{d\vec{L}}{dt} \qquad (i)$$

where, $\vec{\tau} = \vec{r} \times \vec{F}$

is the torque due to external force $\vec{F}$. This simple equation can be complicated when to analyze the resultant motion. A somewhat idealized example is shown below [Fig. (i)].

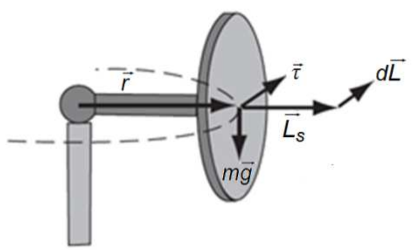

Fig. (i)

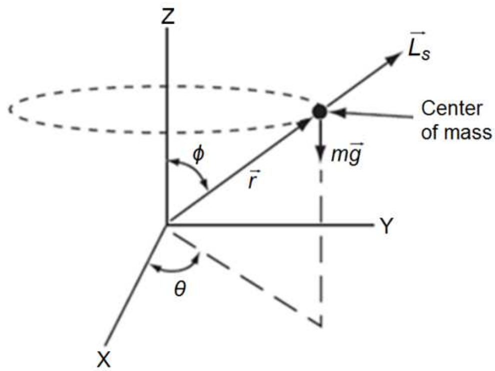

Fig. (ii)

Let us assume that the toy is spinning about its own axis with angular speed $\vec{\omega}$, supported by a frictionless bearing. In this case, gravity will make a torque about the origin as the CM is not above the pivot point. But there are no external forces which can make a torque under frictionless condition. This means that both L_z and ω are constants. Again, the total mechanical energy, in addition to the gravitational potential must be constant. We choose the situation where the top is spinning fast with its axis horizontal. The external force being vertically downwards, the torque is horizontal.

Since the spin angular momentum L_s is parallel to the axis of rotation, from Eq. (i) we can say that dL is perpendicular to L_s, and to a first approximation $L_s + dL$ is the same length as L_s but pointing in a different direction. The tip of L_s traces out a circle at constant angular speed Ω [Fig. (i)]. This motion is said as precession. Let us hold the axis fixed to start the gyroscope and set the rate of spin to the required value. The axis is then shifted at the precession speed and released. The motions will now be a smooth precession. On the other hand, if now the axis is released from the rest position the tip will be able to trace out small looping motions superimposed on overall precession. This is known as nutation which arises owing to conservation of mechanical energy.

Precessional motion is representing extra kinetic energy with reference to the state with the fixed axis. Since ω is constant due to frictionless bearing, this extra KE must comes from a loss of gravitational potential. Thus the centre of mass must fall tipping the rotational axis for the top to precess. When the spin is fast, the drop is low and the precession is affected slightly. In general, the tip of the axis bounces a little up and down and the precessional speed changes slowly.

If a quantitative comparison is made with Eq. (i), the angular frequency Ω of the steady precession can be derived, taking into account that there is no nutation. In that condition we can express the total angular momentum as the sum of the spin and precession angular momenta as,

$$\vec{L} = \vec{L_S} + \vec{L_z} = I\omega(\cos\theta\sin\varphi\hat{i} + \sin\theta\sin\varphi\hat{j} + \cos\varphi\hat{k}) + I_z\Omega\hat{k} \qquad (ii)$$

where I_Z is the moment of inertia about the vertical and the angles are shown in Fig. (ii). For steady precession, φ is constant and $d\theta/dt = \Omega$, hence we can write

$$\frac{d\vec{L}}{dt} = I\omega(-\Omega\sin\theta\sin\varphi\hat{i} + \Omega\cos\theta\sin\varphi\hat{j}) \qquad (iii)$$

The vector from the pivot to the CM is given by,

$$\vec{r} = r\cos\theta\sin\varphi\hat{i} + r\sin\theta\sin\varphi\hat{j} + r\cos\varphi\hat{k} \qquad (iv)$$

Therefore, the gravitational torque is

$$\vec{\tau} = \vec{r} \times m\vec{g} = -rmg\sin\theta\sin\varphi\hat{i} + rmg\cos\theta\sin\varphi\hat{j} \qquad (v)$$

These will satisfy Eq. (i) if

$$\Omega = \frac{rmg}{I\omega} \qquad (vi)$$

It demonstrates that the uniform precession is a possible solution to the equations of motion.

Procedure

Physical arrangement: The gyroscope we will use is a solid metal sphere supported on a cushion of air [Fig. (iii)]. The air cushion supports the sphere under its geometric center. If the sphere was perfect there would be no torques acting at all. Actually the sphere has a rod fastened to it that serves to displace the CM away from the geometric center of the sphere and permit gravity to exert a small torque on the system.

A sliding weight can be fastened to the rod for adjusting the torque as desired. The rod is used conveniently as marker for the rotation axis and this is important for handling, manipulating the gyroscope. There is an availability of secondary jet of air for a rapid spin of the sphere.

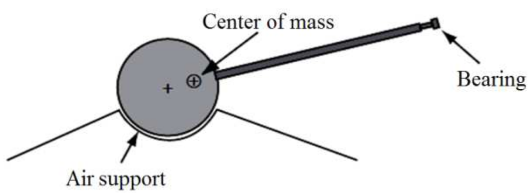

Fig. (iii)

Qualitative Observations: Let us now start the work by turning on the air supply so that the sphere floats. It is essential to lift the sphere slightly to start the flow. If the sliding weight was left on the rod, keep it aside for later use. Holding vertically the rod on the bearing, can twirl the rod between the fingers of other hand to make the sphere spin. We have to do it for a few different angles.

Next try to see the effects of releasing the rod from rest when moving at about the precession speed and when moving in the opposite direction. The precession corresponds to tracing a circle having constant latitude and the nutation is a low value of deviation from the circular path. You have to describe the motion of the tip of the rod for varying conditions by sketching. The slow top motion is distinctive which can be demonstrated spinning the sphere with a vertical rod and then releasing it. Start out with rapid rotation, the motion becomes slow precessions which then slow down the rotation by grasping the rod between fingers. Rate of spin is diminished in several stages, releasing the rod to see the motion after each reduced value. Give a description of the motion.

Quantitative Part of the Experiment: In this experiment, the quantitative part is the checking of two predictions of Eq. (*vi*). We understand that Ω depends on rmg directly while it is inversely depends on ω. As both of these vary, we can verify the prediction. For the experiment we need to spin the sphere very fast and for that we use an air jet. Hold the rod in the horizontal direction, and then direct the air at the equator of the sphere. After a few minutes the sphere will spin rapidly. If we now set the jet aside and thereby release the rod, the gyroscope will precess slowly. One can measure the time for a complete precession using a stopwatch. Rate of spinning will diminish with time.

For measuring the spin velocity we use a strobe light. Position the strobe is so adjusted that it gives bright illumination of the sphere. If the time interval between the flashes is same as the time for one rotation of the sphere, then the rotational motion will appear to be ‘stopped’. This is because the same parts of the sphere will be in the same place at each flash. The sphere is marked with two black lines so that it becomes easier to note when it is stopped. Finding the time interval between flashes, one can determine the period of the spinning motion.

When the strobe is flashing very fast, the sphere will not be able to make a complete revolution between the flashes. We will then observe multiple images of the black markings. On the other hand, when the strobe flashes very slowly one can find a single image only. The flash rate is displayed on the strobe in flashes per minute. We need to convert it to the time interval between flashes, seconds per flash, to obtain the spin period T_s.

To test Eq. (*vi*) we have to determine the precession velocity for a few values of spin velocity. However, the quantities we find directly are the precession and spin periods, denoted as, T_p and T_s. We can then rewrite the Eq. (*vi*) as

$$T_p T_s = \frac{4\pi^2 I}{rmg} \tag{vii}$$

From this we find that the multiplication of T_p and T_s is a constant.

We can verify the dependence of the precession period on rmg, using the sliding weight. An extra weight m', kept on the rod at a distance r' from the centre of the sphere will add an extra term in the denominator of Eq. (*vii*), giving

$$T_p T_s = \frac{4\pi^2 I}{rmg + r'm'g} \tag{viii}$$

It is more informative when both sides are inverted

$$\frac{1}{T_p T_s} = \frac{g}{4\pi^2 I}\left(rm + r'm'\right) \tag{ix}$$

A plot of $(T_p T_s)^{-1}$ vs r' will be a straight line, with an intercept at the value found with no additional weight.

Precautions

Some precautions are to be taken for avoiding problems related to the apparatus. The rod should be sufficiently strong for supporting heavy sphere in the vertical position. The sphere is soft metal and it is exactly rounded for successful performance of the experiment. It will be damaged if falls on the hard floor. It will climb out and damaging itself when the spinning ball contacts the supporting cup which may be avoided by leaving the air supply on whenever the sphere is spinning and not pushing into the support. The rod has a bearing at the end. This may be hold for controlling the sphere without slowing it down. When required to slow the rotation, gently squeeze the rod with two fingers, while holding onto the bearing to steady it.

Viva-Voce Questions

1. What is meant by gyroscope?
2. What is meant by motion of a gyroscope?
3. Give an idealized example of gyroscopic motion.
4. What is meant by precession?
5. What is nutation?
6. Explain the quantitative part of the experiment.
7. What are the precautions to be taken during the experiment?
8. What is your conclusion from the graph of this experiment?

KNOW MORE

Dynamics is the branch of physics in classical mechanics related to the study of forces and their effects. Newton first formulated the fundamental laws of dynamics in classical non-relativistic physics. Foundations of dynamics were laid at the end of the 16th century by Galileo.

The dynamics of the rigid body consists of the observation of the effects of external forces and also the couples on the variation of its six degrees of freedom. The trajectory of any point in the body, taken as reference point, provides the variation of three of these degrees of freedom.

Activity

Euler (1727) articulated some basic ideas which were further extended in 1782 by Giordano Riccati. He developed the method and was able to determine elasticity of some materials, followed by Thomas Young. This study was then expanded by Simeon Poisson to the third dimension with the Poisson ratio. Gabriel Lamé continued the study for assuring stability of structures introducing the Lamé parameters. These coefficients helped to establish linear elasticity theory as well as it began the field of continuum mechanics.

Interesting facts

Leonhard Euler

Leonhard Euler extended all the Newton's laws of motion from the consideration of particles to rigid bodies in addition to two other laws. After Newton, many scientists actively participated in the re-formulations progressively allowing solution to many problems. Out of these, the first one was developed in 1788 by Joseph Louis Lagrange, an Italian-French mathematician. In the Lagrangian mechanics the solution considered the path of least action following the calculus of variations. William Rowan Hamilton again re-formulated Lagrangian mechanics in 1833.

The Hamiltonian mechanics allowed a more in-depth look at the principles. In fact, a majority of Hamiltonian mechanics can be found in quantum mechanics. Computer simulators are almost risk free method to train people for handling heavy equipment in critical situations under large stress to follow a curved path. Examples are large forest machines, bulldozers to cable simulations in tug boats and maneuver belt vehicles. Driving simulators is also a good example. The simulators need to be responsive and sufficiently accurate to give proper predictions of the virtual equipment being handled. This is analogous to interactive virtual prototyping. The driving simulators are very specialized and include a number of aspects of virtual reality. The largest and most complex simulator today is the National Advanced Driving Simulator (NADS).

Analogy

Dynamics includes kinematics, related to the geometry of motion. Kinematics can be used to relate displacement to velocity to acceleration to time without any reference to originate the motion. Kinetics predicts the motion by given forces or to find the forces to produce a given motion. In general, dynamics is the branch of mechanics which deals with the effect on the motion of the objects. On the contrary, statics deals with the study of forces without motion.

With the successful implementation of the dynamics, one can get the ability to calculate dynamic loads in operating structures and in their members in 2 and 3 dimensions. The ability to design and implement trajectories by the members of a structure and to measure the velocities and accelerations at any point of the structure in 2D and 3D becomes possible. In addition, the ability to combine kinematic and dynamic analysis at every step of the operating envelop, regardless of the potential problem approach as well as awareness of the fact that the dynamic loads can be calculated using the assumption of rigid body helps to design the dimensions properly. In fact, this knowledge is used in many subsequent courses of Mechanical and Aeronautical Engineering.

History

The foundations of dynamics were laid at the end of the 16th century when Galileo Galilei, experimenting with a smooth ball rolling down an inclined plane, derived the law of motion for falling bodies. He was also the first to show that the force is the cause of changes in the velocity of a body. Newton's 'Principia' (1687) stands among the world's greatest scientific and intellectual achievements but it does not address all the general principles of mechanics. In the book there is no theory of general dynamical systems for rigid bodies and also nothing on the mechanics of deformable solid and fluid continuum. For mass points Newton's theory is just insufficiently general for delivering a unifying method.

More than half a century of research with solutions of special problems, on the general principles of mechanics applicable to all bodies, was established by Euler in 1750 and thereafter.

Timelines

1690: *James Bernoulli* established that the cycloid is the solution to the tautochrone problem.

1727: *Leonhard Euler* extended Newton's laws of motion from particles to rigid bodies with two additional laws.

1747: *D'Alembert* and *Alexis Clairaut* published first approximate solutions to the three-body problem.

1782: *Giordano Riccati* started to find elasticity of some materials; it was followed by *Thomas Young* and *Simeon Poisson* who expanded study to the third dimension; *Gabriel Lamé* drew on the study for assuring stability of structures and introduced the Lamé parameters.

1828: *Gauss* formulated the Principle of Least Constraint to incorporate the matter of constraints on motion into the principles of physical dynamics.

1843: *Arthur Cayley* systems of n-variables.

1983: *Mordehai Milgrom* proposed modified Newtonian dynamics.

Applications (Real Life / Industrial)

The advantage of Hamiltonian mechanics was that its framework permitted a more in-depth look. Most of the framework of Hamiltonian mechanics can be found in quantum mechanics. Some difficulties were noted in the late 19th century which could only be resolved by modern physics. When combined with classical thermodynamics, classical mechanics leads to the Gibbs paradox where entropy is not a well-defined quantity. The effort for resolving these problems led to the development of quantum mechanics.

In a similar way, the different behavior of classical electromagnetism and classical mechanics under velocity transformations led to the theory of relativity. Some of the important areas of applications are: for the analysis of robotic systems, for the biomechanical analysis of animals, humans or humanoid systems, for the analysis of space objects, for the understanding of strange motions of rigid bodies, for the design and development of dynamics-based sensors, such as gyroscopic sensors, for the design and development of various stability enhancement applications in automobiles for improving the graphics of video games which involves rigid bodies etc.

Case Study (Environmental / Sustainability / Social / Ethical Issues)

Classical mechanics is the mathematical science to study the displacement of bodies under the action of forces. Newton established that mathematical approach with a view to analyze physical phenomena wherein he stated that it was not required to introduce hypothesis having no experimental basis. This led to an application of the principles of Newtonian mechanics to many new problems culminating in the study of Euler who began a systematic study of the three dimensional motion of rigid bodies leading to a set of dynamical equations called Euler's equations of motion.

During this time the experimental methods and mathematical tools of Newtonian mechanics were applied to many non-rigid systems of particles leading to the development of continuum mechanics. The theories of fluid mechanics, wave mechanics and electromagnetism emerged leading to the development of the wave theory of light. The fundamental concepts of absolute time and space, as Newton defined in the

Principia, were inadequate to explain a host of experimental findings. Einstein, rethinking the concepts of space and time as well as the relativity of motion, resolved the apparent conflicts between optics and Newtonian mechanics.

The emerging technology made it possible to use rigid body simulators as sub-parts in larger systems. For instance, in time critical scenarios like tracking humans or maneuvering a robot, a simulator can be conveniently used as a prediction tool. From a digital design viewpoint, we may define a spectrum of technology where at one end we find off-line simulators that may take hours or days to compute results while on the other hand they deliver high quality results. For movie production several similar computer graphics simulation methods can be utilized conveniently. If we look at the spectrum we find the fast run-time simulators capable of delivering plausible results too fast. This type of simulator is originated from game physics, *e.g.*, Bullet.

Inquisitiveness and Curiosity Topics

For games and movies rigid body simulation has to be plausible rather than physically realistic. For games the simulation needs real-time but for movies do not have the real-time constraint. However, fast simulation methods are always preferred, as complex scenarios are simulated for special effects and simulation time costs money. Iterative constraint solving methods are very popular in both the areas. Many games using 2D and 3D graphics rely on a rigid body dynamics engine for dealing with collision detection and collision response. In some situations the motion of the objects is fully driven by rigid body dynamics. As for example, we can mention the game Angry Birds, using the Box2D physics engine or a 3D Jenga game. This further introduces a big challenge of interaction between rigid bodies and kinematically animated objects.

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TABLE OF PHYSICAL CONSTANTS

Table I: Units and dimensions

Physical Quantities		Dimension		CGS System		SI System	
		Dimension	Unit	Name	Unit Symbol	Name	Unit Symbol
Fundamental quantities	Length	[L]	l	centimetre	cm	metre	m
	Mass	[M]	m	gram	g	kilogram	kg
	Time	[T]	t	second	s	second	s
Derived quantities	Area	$[L^2]$	S, A	square centimetre	cm^2	square metre	m^2
	Volume	$[L^3]$	V	cubic centimetre	cm^3	cubic metre	m^3
	Velocity	[L]/[T]	v	centimetre/ second	cm/s	metre/ second	m/s
	Acceleration	$[L]/[T^2]$	a	centimetre/ square second	cm/s^2	metre/ square second	m/s^2
	Force	$[M][L]/[T^2]$	F	Dyne	dyn = $g.cm/s^2$	Newton	N = $kg.m/s^2$
	Energy	$[M][L^2]/[T^2]$	E	Erg	erg = $g.cm^2/s^2$	Joule	J = $kg.m^2/s^2$

Table II: Powers of ten

Prefix	Abbreviation	Power	Prefix	Abbreviation	Power
Deka	dek	10^1	deci	d	10^{-1}
Hecto	h	10^2	centi	c	10^{-2}
Kilo	k	10^3	milli	m	10^{-3}
Mega	M	10^6	micro	μ	10^{-6}
Giga	G	10^9	nano	n	10^{-9}
Tera	T	10^{12}	pico	p	10^{-12}
Peta	P	10^{15}	femto	f	10^{-15}
Exa	E	10^{18}	atto	a	10^{-18}

Table III: Coefficient of static friction for dry interfaces

Material combination	Coefficient of static friction
Rubber on Concrete	0.60 – 0.90
Stone on Stone	0.40 – 0.70
Metal on Metal	0.15 – 0.60
Metal on Wood	0.20 – 0.60
Wood on Wood	0.25 – 0.50
Metal on Stone	0.30 – 0.70
Wood on Leather	0.25 – 0.50
Metal on Leather	0.30 – 0.60

APPENDICES

APPENDIX-A: Suggestive Template for Practicals

- **Aim**
  Explain briefly about the aim of the experiment.
- **Relevance**
  Explain about the relevancy of the experiment in your own words.
- **Requirements**
  List out all the required apparatus along with their proper specifications.
- **Procedure, Observations and Inference**
  Explain the procedure of the experiment step-wise and note the observations properly. On the basis of observations certain inference is to be made. You can use a table similar to that given below:

Step No.	Procedure	Observation	Inference
1			
2			
3			

- **Video / animation**
  If possible, you can go through some video/animation to visualize the steps physically.
- **Calculations**
  Properly calculate all the required physical quantities essential for your experiment.
- **Result and Discussion (Error measurement)**
  Obtain the final result and discuss about it with proper considerations of errors which can be introduced during your experiment.
- **Conclusions**
  Finally give your conclusion based on the obtained results.
- **Validation of the topics in Experiment**
  Try to validate the result of the experiment in real life scenario.
- **Use of ICT**
  You can also study using the available online resources. These are useful as there is no time constraint at all. Some of which are listed (not limited to) below:
  https://swayam.gov.in/
  https://nptel.ac.in/
  https://www.swayamprabha.gov.in/

Note for Instructor and Lab-Technicians
Some general and specific instructions are listed separately [see Annexure V] for laboratory preparation, maintenance, safety aspects, etc. Laboratory Instructor and Lab-Technicians can follow those instructions properly to run the laboratory smoothly without any hazard.

APPENDIX-B: Indicative Evaluation Guidelines for Practicals / Projects / Activities in Group

Process Related Skills

Criteria and Level	Developing	Competent	Proficient
Handling the Set-up			
Recording of Data			
Time management			
Team Work			
Individual Work			
Safety Precautions			

Product Related Skills

Criteria and Level	Developing	Competent	Proficient
Content			
Research/Survey			
Use of latest Technology			
Stays on Topic			
Preparedness			
Confidence of Presentation			
ICT Usage including ppt Making Skill			
Time Management			
Group Efforts			
Individual Efforts			

APPENDIX-C: Assessments Aligned to Bloom's Level

- Bloom's Taxonomy – It has been coupled into following two categories for development of Questions for this Book as given below:

Category I Questions	Category II Questions *- Higher Order Thinking Skills*
Bloom's Level 1: Remember Bloom's Level 2: Understand Bloom's Level 3: Apply	Bloom's Level 4: Analyse Bloom's Level 5: Evaluate Bloom's Level 6: Create

APPENDIX-D: Records for Practicals

Sl No.	Page No.	Name of the Experiment	Date			Marks	Signature
			Actual	Repeat	Remarks		
1.		Experiments on an air-track					
2.		To investigate the resonance phenomena in mechanical oscillators					
3.		To investigate the normal modes in coupled oscillations of two bodies					
4.		Determination of the moment of inertia of a flywheel					
5.		Determination of the moment of inertia of a rectangular bar					
6.		To qualitatively verify typical motions of a gyroscope					

ANNEXURES

Annexure- I: Important formulas involving vector differential operator

(i) $\vec{\nabla} \times \vec{\nabla} \times \vec{F} = \vec{\nabla}\left(\vec{\nabla}.\vec{F}\right) - \nabla^2 \vec{F}$

(ii) $\vec{\nabla}.\left(\vec{\nabla} \times \vec{F}\right) = 0$ *i.e.*, the divergence of a curl is always zero.

(iii) $\vec{\nabla} \times \left(\vec{\nabla}\phi\right) = 0$ *i.e.*, the curl of a gradient is always zero.

(iv) $\vec{\nabla} \times \left(\vec{F} \times \vec{G}\right) = \left(\vec{G}.\vec{\nabla}\right)\vec{F} - \left(\vec{F}.\vec{\nabla}\right)\vec{G} + \left(\vec{\nabla}.\vec{G}\right)\vec{F} - \left(\vec{\nabla}.\vec{F}\right)\vec{G}$

(v) $\vec{\nabla}\left(\vec{F}.\vec{G}\right) = \left(\vec{F}.\vec{\nabla}\right)\vec{G} + \left(\vec{G}.\vec{\nabla}\right)\vec{F} + \vec{F} \times \left(\vec{\nabla} \times \vec{G}\right) + \vec{G} \times \left(\vec{\nabla} \times \vec{F}\right)$

(vi) $\vec{\nabla}.\left(\vec{F} \times \vec{G}\right) = \vec{G}.\left(\vec{\nabla} \times \vec{F}\right) - \vec{F}.\left(\vec{\nabla} \times \vec{G}\right)$

(vii) $\vec{\nabla}.\left(\phi\vec{F}\right) = \left(\vec{\nabla}\phi\right).\vec{F} + \phi\left(\vec{\nabla}.\vec{F}\right)$

(viii) $\nabla^2 V = \vec{\nabla}.\vec{\nabla}V \neq \vec{\nabla}\left(\vec{\nabla}.V\right)$

(ix) $\vec{\nabla}\left(U + V\right) = \vec{\nabla}U + \vec{\nabla}V$

(x) $\vec{\nabla}\left(UV\right) = V\vec{\nabla}U + U\vec{\nabla}V$

(xi) $\vec{\nabla}\left(\frac{U}{V}\right) = \frac{V\vec{\nabla}U - U\vec{\nabla}V}{V^2}$

(xii) $\vec{\nabla}V^n = nV^{n-1}\vec{\nabla}V$

(xiii) $\vec{\nabla}\left(f\vec{A}\right) = f\left(\vec{\nabla}.\vec{A}\right) + \vec{A}.\left(\vec{\nabla}f\right)$

(xiv) $\vec{\nabla} \times \left(f\vec{A}\right) = f\left(\vec{\nabla} \times \vec{A}\right) - \vec{A} \times \left(\vec{\nabla}f\right)$

Annexure- II: Triple product of vectors

Triple product of three vectors may be classified as: i) Scalar and ii) vector triple product

Scalar triple product: The scalar triple product of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ may be defined as:

$$\vec{A}.\left(\vec{B}\times\vec{C}\right)=\vec{B}.\left(\vec{C}\times\vec{A}\right)=\vec{C}.\left(\vec{A}\times\vec{B}\right)$$

The scalar triple product of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ basically represents the volume of a parallelepiped whose adjacent sides are $\vec{A}$, $\vec{B}$ and $\vec{C}$ and is given by,

$$V=\left|\vec{A}.\left(\vec{B}\times\vec{C}\right)\right|=\left|\vec{B}.\left(\vec{C}\times\vec{A}\right)\right|=\left|\vec{C}.\left(\vec{A}\times\vec{B}\right)\right|=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

It may be positive or negative

Vector triple product: The vector triple product of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ may be defined as:

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}.\vec{C}\right)-\vec{C}\left(\vec{A}.\vec{B}\right)$$

It is a vector coplanar with $\vec{B}$ and $\vec{C}$. Generally, $\vec{A}\times\left(\vec{B}\times\vec{C}\right)\neq\left(\vec{A}\times\vec{B}\right)\times\vec{C}$

The following laws are valid for dot product of two vectors:

1. $\vec{A}.\vec{B} = \vec{B}.\vec{A}$ *Commutative law for dot products*
2. $\vec{A}.\left(\vec{B}+\vec{C}\right) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$ *Distributive law*
3. $m(\vec{A}.\vec{B}) = (m\vec{A}).\vec{B} = \vec{A}.(m\vec{B}) = (\vec{A}.\vec{B})m$; where *m* is a scalar.
4. $\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1,\ \hat{i}.\hat{j}=\hat{j}.\hat{i}=\hat{i}.\hat{k}=\hat{k}.\hat{i}=\hat{j}.\hat{k}=\hat{k}.\hat{j}=0$
5. If $\vec{A} = A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$ and $\vec{B} = B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$ then $\vec{A}.\vec{B} = A_xB_x + A_yB_y + A_zB_z$

If $\vec{A}.\vec{B} = 0$ and $\vec{A}$ and $\vec{B}$ are not null vectors, then $\vec{A}$ and $\vec{B}$ are perpendicular.
The following laws are valid for cross product of two vectors:

1. $\vec{A}\times\left(\vec{B}+\vec{C}\right) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$ *Distributive law*
2. $m(\vec{A}\times\vec{B}) = (m\vec{A})\times\vec{B} = \vec{A}\times(m\vec{B}) = (\vec{A}\times\vec{B})m$; where *m* is a scalar.
3. $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$, $\hat{i}\times\hat{j}=-\hat{j}\times\hat{i}=\hat{k}$; $\hat{j}\times\hat{k}=-\hat{k}\times\hat{j}=\hat{i}$ and $\hat{k}\times\hat{i}=-\hat{i}\times\hat{k}=\hat{j}$
4. If $\vec{A} = A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$ and $\vec{B} = B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$ then

$$\vec{A}\times\vec{B}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}=\hat{i}(A_yB_z-A_zB_y)+\hat{j}(A_zB_x-A_xB_z)+\hat{k}(A_xB_y-A_yB_z)$$

If $\vec{A}\times\vec{B} = 0$ and $\vec{A}$ and $\vec{B}$ are not null vectors, then $\vec{A}$ and $\vec{B}$ are parallel to each other.

Annexure- III: Gradient, divergence and curl in Cartesian, cylindrical and spherical polar coordinate system

Gradient: General Expression

$$\vec{\nabla}V = \frac{1}{h_1}\frac{\partial V}{\partial u}\hat{a}_u + \frac{1}{h_2}\frac{\partial V}{\partial v}\hat{a}_v + \frac{1}{h_3}\frac{\partial V}{\partial w}\hat{a}_w$$

For the Cartesian, cylindrical and spherical polar coordinate system, the expressions for gradient can be written as:

In Cartesian coordinates:

$$\vec{\nabla}V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z$$

In cylindrical coordinates:

$$\vec{\nabla}V = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\varphi}\frac{\partial V}{\partial \varphi}\hat{a}_\varphi + \frac{\partial V}{\partial z}\hat{a}_z$$

and in spherical polar coordinates:

$$\vec{\nabla}V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{a}_\varphi$$

Divergence: General Expression

$$\vec{\nabla}.\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial\left(h_2 h_3 A_u\right)}{\partial u} + \frac{\partial\left(h_1 h_3 A_v\right)}{\partial v} + \frac{\partial\left(h_1 h_2 A_w\right)}{\partial w}\right]$$

For the Cartesian, cylindrical and spherical polar coordinate system, the expressions for divergence can be written as:

In Cartesian coordinates:

$$\vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

In cylindrical coordinates:

$$\vec{\nabla}.\vec{A} = \frac{1}{\rho}\frac{\partial\left(\rho A_\rho\right)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

In spherical polar coordinates:

$$\vec{\nabla}.\vec{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta A_\theta\right)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

Curl: General Expression

$$\vec{\nabla}\times\vec{A}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{a}_u & h_2\hat{a}_v & h_3\hat{a}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1A_u & h_2A_v & h_3A_w \end{vmatrix}$$

In Cartesian coordinates:

$$\vec{\nabla}\times\vec{A}=\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

In cylindrical coordinates,

$$\vec{\nabla}\times\vec{A}=\frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\varphi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}$$

In spherical polar coordinates,

$$\vec{\nabla}\times\vec{A}=\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix}$$

Annexure- IV: Vector integration - line, surface and volume integrals

Line Integral: Line integral $\int_C \vec{E}.d\vec{l}$ is the dot product of a vector with a specified *C*; in other words it is the integral of the tangential component $\vec{E}$ along the curve *C*.

As shown in the Fig. A.1, given a vector $\vec{E}$ around *C*, we define the integral $\int_C \vec{E}.d\vec{l} = \int_a^b Edl\cos\theta$ as the line integral of *E* along the curve *C*.

If the path of integration is a closed path as shown in the figure the line integral becomes a closed line integral and is called the circulation of $\vec{E}$ around *C* and denoted as $\oint_C \vec{E}.d\vec{l}$ as shown in the Fig. A.2.

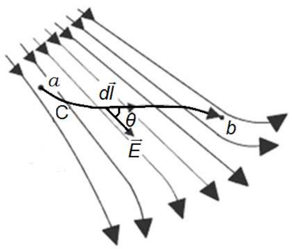

Fig. A.1: Line Integral

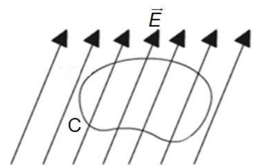

Fig. A.2: Closed line Integral

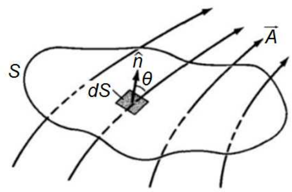

Fig. A.3: Surface Integral

Surface Integral: Given a vector field $\vec{A}$, continuous in a region containing the smooth surface *S*, we define the surface integral or the flux of $\vec{A}$ through *S* as

$$\varphi = \int_S A\cos\theta dS = \int_S \vec{A}.\hat{n}dS = \int_S \vec{A}.d\vec{S}$$

as surface integral over surface *S*.

If the surface integral is carried out over a closed surface, then we write

$$\varphi = \oint_S \vec{A}.d\vec{S}$$

Volume Integral: We define $\iiint_V fdV$ as the volume integral of the scalar function f(function of spatial coordinates) over the volume *V*. Evaluation of integral of the form $\iiint_V fdV$ can be carried out as a sum of three scalar volume integrals, where each scalar volume integral is a component of the vector $\vec{f}$.

Annexure- V: Different type of errors in measurements

Measurement and its importance in Science and Technology

Measurements represent quantities relating to a real time system by making use of numerical values. Requirements for accurate measurements are:

i) Apparatus must be accurate,

ii) Method used ought to be provable and

iii) Standard used should be defined correctly.

In science and technology, advancement is of little significance without the availability of actual measured values with practical proofs. A device to determine variable is known as an instrument. It serves an aid for humans to measure values of unknown quantities. An instrument can be electronic, mechanical or electrical. Based on the degree of variation of the measured quantity with respect to time, an instrument can have static or dynamic characteristics.

Errors in Measurement

If ideal conditions are applied for measuring any parameter, the average deviations of different factors tend to be zero. Average of these infinite numbers of measured values is known as True Value. But this is hypothetical as the negative and positive deviations do not cancel each other in practice. The measured value obtained under ideal conditions is considered as the true value or the best-measured value. A difference between the actual value and the true value is called an Error.

Types of Error

Systematic errors are broadly of three categories;

i) Instrumental Error,

ii) Environmental Error and

iii) Observational Error.

Instrumental errors occur owing to shortcomings in the instruments, improper use of instruments or loading effect of the instrument. Improper construction, calibration or operation of an instrument might result in some inherent errors. Environmental errors occur due to external ambient conditions of the instrument which include changes in temperature, humidity, availability of dust, vibrations or effects of external magnetic or electrostatic fields. Random errors occur due to some small factors which fluctuate from one measurement to another.

The manufacturer of any instrument defines a certain accuracy that depends on the type of material and the effort to manufacture the instrument. This accuracy is defined within a certain deviations from the nominal value. The limits of these deviations are termed as Limiting Errors. The ratio of error to the specified nominal value is called relative Limiting Error. Manual errors in reading an instrument or recording any measurement are known as Gross errors which occur during the experiments. The errors in any scientific measurement may happen from different sources which are presented in Fig. A.4. Observational errors may occur due to the fault study of the instrument reading for many sources. Environmental errors, on the other hand, happen due to the outside situation of the measuring instruments, mostly due to the temperature result, force, moisture, dirt, vibration.

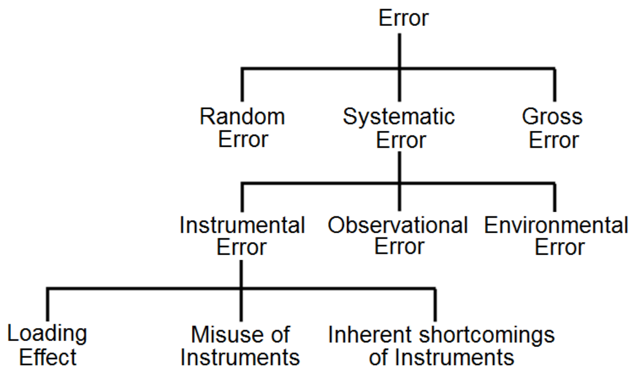

Fig. A.4: Error in measurements

There is an inherent limitation of devices due to their mechanical arrangement.

Calculation of Error in Measurement

These errors are categorized into three types:

i) absolute error,

ii) relative error, and

iii) percentage error.

The absolute error is defined as the variation between the values of actual and measured quantities.

If we denote the measured value as V_A, and the exact value as V_E, then we have

$$\text{Absolute error} = |V_A - V_E|$$

$$\text{Relative Error} = \frac{\text{Absolute error}}{\text{Actual error}}$$

$$= \frac{|V_A - V_E|}{V_E}$$

$$\text{Percentage error (\%)} = \frac{|V_A - V_E|}{V_E} \times 100$$

Example: A length was calculated as 6.8 cm but the absolute length was 6.74 cm. Find the absolute, relative and percentage errors.

Ans: Given that $V_A = 6.8$ cm and $V_E = 6.74$ cm

$$\text{Absolute error} = |V_A - V_E| = |6.8 - 6.74| = 0.06 \text{ cm}$$

$$\text{Relative Error} = \frac{|V_A - V_E|}{V_E} = \frac{0.06}{6.74} = 0.0089$$

$$\text{Percentage error (\%)} = \frac{|V_A - V_E|}{V_E} \times 100 = \frac{0.06}{6.74} \times 100 = 0.89\%$$

Arithmetic Mean Value

To minimize random errors, the measurements are repeated and the average value is taken as the correct value of the measured quantity. The arithmetic mean value would be very close to the most accurate reading. If the number of observation is taken 'n' times, the random error reduces to $\frac{1}{n}$ times.

Let a_1, a_2, $a_{3\dots}$, a_n are the n different measured readings of a physical quantity. The most accurate value is its arithmetic mean value which can be obtained from,

$$a_{mean} = \frac{a_1 + a_2 + \dots\dots + a_n}{n} = \frac{1}{n}\sum_{i=1}^{n} a_i$$

Absolute Error

The magnitude of the difference between the true value of the quantity and the measured value is known as the absolute error in the measurement. Since the true value of the quantity is not known, the arithmetic mean of the measured values is taken as the true value.

If a_1, a_2,, ….. are the measured values of a certain quantity, the errors Δa_1, Δa_2, ……… in the measurements are

$$\Delta a_1 = a_{\text{mean}} - a_1$$

$$\Delta a_2 = a_{\text{mean}} - a_2$$

$$\Delta a_3 = a_{\text{mean}} - a_3$$

$$\Delta a_4 = a_{\text{mean}} - a_4 \qquad \text{and so on}$$

The arithmetic mean of all the absolute errors is considered as the final absolute error in the measurement and is called mean absolute error. The value obtained in a single measurement may be in the range: $a_{\text{mean}} \pm \Delta a_{\text{mean}}$

Relative Error

The ratio of the absolute error to the true value of the measured quantity is known as the relative error or fractional error. As the arithmetic mean value is taken as the true value, the relative error can be expressed as,

$$\text{Relative error, } \delta a = \frac{\Delta a_{mean}}{a_{mean}}$$

Percentage Error

It is the relative error when expressed in percentage. Thus, we have,

$$\text{Percentage error } = \frac{\Delta a_{mean}}{a_{mean}} \times 100\%$$

Example: Using a screw gauge successive measured readings of the radius of a wire are 1.21 mm, 1.19 mm, 1.20 mm, 1.18 mm and 1.17 mm respectively. Determine the absolute errors and the relative error in the measurement.

Ans: Arithmetic mean value is, $a_{mean} = \dfrac{1.21 + 1.19 + 1.20 + 1.18 + 1.17}{5} = 1.19\,\text{mm}$

Table A.1: Error in measurements

Difference between a_{mean} and measured value in mm	Magnitude of error in mm
1.21–1.19 = 0.02	0.02
1.19–1.19 = 0.00	0.00
1.20–1.19 = 0.01	0.01
1.18–1.19 = –0.01	0.01
1.17–1.19 = –0.02	0.02

The arithmetic mean of the absolute errors is, $\Delta a_{mean} = \dfrac{0.02 + 0.00 + 0.01 + 0.01 + 0.02}{5} = 0.016\,\text{mm}$

Relative error, $\delta a = \dfrac{\Delta a_{mean}}{a_{mean}} = \dfrac{0.016}{1.19} = 0.0134$

Percentage error $= \dfrac{\Delta a_{mean}}{a_{mean}} \times 100\% = \pm 1.34\%$

Combination of errors

If a quantity is obtained by combining a few measurements, the errors in those measurements can be combined in some way or other.

Error in the sum of the quantities

We consider two quantities A and B which have measured values $A \pm \Delta A$ and $B \pm \Delta B$ respectively; ΔA and ΔB are the absolute errors in the measurements.

To determine the error ΔZ which may occur in the sum $Z = A + B$, we consider

$$Z \pm \Delta Z = (A \pm \Delta A) + (B \pm \Delta B) = A + B \pm \Delta A \pm \Delta B$$

Maximum possible error in the value of Z is,

$$\Delta Z = \Delta A + \Delta B.$$

Hence when two quantities are added, the absolute error in the result is the sum of the absolute errors in the measured quantities.

Error in the difference of the quantities

We consider two quantities, A and B which have measured values $A \pm \Delta A$ and $B \pm \Delta B$ respectively; ΔA and ΔB are the absolute errors in their measurements.

To determine the error ΔZ which may occur in the difference $Z = A - B$, we consider

$$Z \pm \Delta Z = (A \pm \Delta A) - (B \pm \Delta B) = A - B \pm \Delta A \pm \Delta B$$

Hence the maximum possible error in the value of Z is given by

$$\Delta Z = \Delta A + \Delta B.$$

So when two quantities are subtracted, the absolute error in the result is the sum of the absolute errors in the measured quantities.

Error in the product of the quantities

We consider two quantities, A and B which have measured values $A \pm \Delta A$ and $B \pm \Delta B$ respectively; ΔA and ΔB are the absolute errors in their measurements.

To get the error ΔZ which may occur in the product $Z = AB$, we consider

$$Z \pm \Delta Z = (A \pm \Delta A)\,(B \pm \Delta B) = AB \pm A\Delta B \pm B\Delta A \pm \Delta A\Delta B$$

Dividing left side by Z and right side by AB we have,

$$1 \pm \frac{\Delta Z}{Z} = 1 \pm \frac{\Delta A}{A} \pm \frac{\Delta B}{B} \pm \frac{\Delta A\Delta B}{AB}$$

The maximum fractional error in Z is,

$$\frac{\Delta Z}{Z} = \frac{\Delta A}{A} \pm \frac{\Delta B}{B}$$

(As ΔA and ΔB are small, their products $\frac{\Delta A\Delta B}{AB}$ are too small and can be ignored)

Hence when two quantities are multiplied, the fractional error in the result is the sum of the fractional errors in the measured quantities.

Error in the quotient of the quantities

We consider two quantities, A and B which have measured values $A \pm \Delta A$ and $B \pm \Delta B$ respectively; ΔA and ΔB are the absolute errors in their measurements.

To find the error ΔZ that may occur in the quotient $Z = \frac{A}{B}$, we consider

$$Z \pm \Delta Z = \frac{A \pm \Delta A}{B \pm \Delta B}$$

In a similar way, solving we have, the maximum fractional error in Z as,

$$\frac{\Delta Z}{Z} = \frac{\Delta A}{A} \pm \frac{\Delta B}{B}$$

Hence when two quantities are divided, the fractional error in the result is the sum of the fractional errors in the measured quantities.

It shows that the maximum percentage error in Z is the sum of the maximum percentage error in A and maximum percentage error in B *i.e.*,

$$\frac{\Delta Z}{Z} \times 100 = \frac{\Delta A}{A} \times 100 + \frac{\Delta B}{B} \times 100$$

Error when a quantity is raised to a power

The error ΔZ which may occur when a quantity is raised to its n^{th} power is n times the fractional error in the quantity itself *i.e.*, if $Z = A^n$,

$$\frac{\Delta Z}{Z} = n\frac{\Delta A}{A}$$

The maximum percentage error in Z can be written as,

$$\frac{\Delta Z}{Z} \times 100 = n \times \frac{\Delta A}{A}$$

Systematic Errors

Systematic errors occur due to fault in the measuring device. They are also called as Zero Error – a positive or negative error. These errors can be detached after correction of the measurement device. These errors are classified into different categories.

Systematic errors are categorized as:

- Instrumental error
- Environmental error
- Observational error
- Theoretical error

Parallax Error

This error occurs due to wrong observations of reading in the instruments. The wrong observations may be due to parallax. To minimize the parallax error highly accurate meters provided with mirror scales are essential.

Estimating Random Errors

There are many ways to make an estimate of the random error in a particular measurement. The simplest way is to make a series of measurements of a given quantity (say, x) and calculate the mean and standard deviation ($\bar{x}$ and σ_x) from this data.

The mean $\bar{x}$ is defined as,

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{n} x_i$$

where, x_i is the result of the i^{th} measurements, N is the number of measurements

The standard variation is given by,

$$\sigma_x = \frac{1}{N}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$$

Annexure-VI: Some general and specific instructions when working in the laboratory

General Instructions

1. In the laboratory, work quietly and cautiously. Remember the main purpose of doing any experiment is to make faithful measurements.
2. Always share equally all the steps of the work with your partner.
3. Presentations of data in tabular form, graphs and calculations should be done correctly and sincerely.
4. Be always honest at the time of recording and representing the experimental data.
5. It is very important to keep in mind that never make up readings or doctor them to get a better fit of the graph as per theory. If any reading appears incorrect, you have to repeat the measurement again and again to find the source of error.
6. At the time of drawing the graph all the data obtained from experiment are to be properly plotted.
7. It is a fact that the objective of the laboratory is learning and also a verification of the knowledge that you have gathered. The experiments are designed properly for the purpose of illustrating different phenomena in all the important areas of Physics.
8. By doing the experiment with your own interest only it is possible to be familiar with all the fine points and to expose you to measuring instruments.
9. Always perform the experiment with an attitude of learning and with your interest to verify the theoretical knowledge that you have gathered.
10. Be very particular to arrive in time in the laboratory and always with proper preparation with a clear knowledge about the experiment.

Specific Instructions

1. When working in the laboratory for collecting data of your experiment, it is important to note all the measured data neatly in the notebook.
2. The recorded data entered in the notebook have to confirm by your instructor before leaving the laboratory.
3. All the students doing the same experiment have to maintain individual copy of the recorded data. The laboratory notebook is required to bring in the laboratory regularly when you come for doing the experiment.
4. Graphs are to be drawn properly at the end of each of experiment.
5. For this you need to know how to optimize on usage of graph paper. Remember all the repeated data are to be accommodated on a single graph sheet.
6. Graphs are to be labeled properly along with the axes showing the corresponding units.
7. During the working hours in the laboratory you are supposed to fully utilize the duration and do not leave the laboratory before the completion of the working hours. If you finish early, you may spend the remaining time to complete the calculations and graphs drawing and for that in the laboratory you are supposed to come equipped with calculators, pencils and scale.

REFERENCES FOR FURTHER LEARNING

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CO AND PO ATTAINMENT TABLE

Course outcomes (COs) for this course can be mapped with the programme outcomes (POs) after the completion of the course and a correlation can be made for the attainment of POs to analyze the gap. After proper analysis of the gap in the attainment of POs necessary measures can be taken to overcome the gaps.

Table for CO and PO attainment

Course Outcomes	Attainment of Programme Outcomes *(1- Weak Correlation; 2- Medium correlation; 3- Strong Correlation)*											
	PO-1	**PO-2**	**PO-3**	**PO-4**	**PO-5**	**PO-6**	**PO-7**	**PO-8**	**PO-9**	**PO-10**	**PO-11**	**PO-12**
CO-1												
CO-2												
CO-3												
CO-4												
CO-5												
CO-6												

The data filled in the above table can be used for gap analysis.

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