K. K. Wagh Institute of Engineering Education and Research, Nashik

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Model Answer (Set 1)

End-Sem Examination, Winter 2025

Academic Year: 2025-2026	Semester: I
Class: PG-I	Program: Structural Engineering
Branch Code: CIV	Pattern: 2024
Name of Course: Theory of Plates and Shells	Course Code: 2404511

Q.1a

a. Explain the distinction between thin and thick plates and state the assumptions that are made in thin plates with small deflections.

the following assumptions are made:

1. Thickness is small compared to other dimensions: The plate thickness is much smaller than its length and width Straight normals remain straight and normal: Lines normal to the mid-surface before deformation remain straight and normal to the mid-surface after deformation (i.e., no transverse shear deformation is considered).
2. Normal stress in thickness direction is negligible: The stress component perpendicular to the plate surface (σ_z) is assumed to be zero.
3. Deflections are small: Transverse deflections are much smaller than the thickness of the plate; hence, geometric nonlinearity is ignored.
4. Plane sections remain plane: Plane cross-sections before deformation remain plane after deformation.
5. Material is linearly elastic and isotropic: The plate material obeys Hooke's law and behaves identically in all directions.

Difference between Thin and Thick Plates:

Aspect	Thin Plates	Thick Plates
Thickness (t)	Small (typically t<L10t < \frac{L}{10}t<10L)	Large (typically t>L10t > \frac{L}{10}t>10L)
Theory Used	Kirchhoff-Love theory (Classical Plate Theory)	Mindlin-Reissner theory (First-order shear deformation theory)
Shear Deformation	Neglected	Considered
Normal Assumption	Normals remain straight and normal	Normals remain straight but not necessarily normal
Stress Distribution	Linear across thickness	Non-linear due to shear effects

Accuracy	Good for slender plates	Necessary for short, thick plates
Applications	Floors, slabs, shells	Machine parts, composite panels

Q.2 a

Describe the underlying presumptions of Levy's solutions for rectangular plates with at least two opposing sides that are simply supported while carrying a load that is uniformly distributed.

2.4.1 Levy's solution for rectangular plate with at least two opposite edges simply supported carrying a uniformly distributed load.

Assumptions:

1. M Levy assumed that two opposite edges are simply supported and other two edges with arbitrary supports or any type of supports.
2. It is assumed that the sides $x = 0$ and $x = a$ are simply supported.
3. It is assumed that the whole load q is shared along x direction producing deflection $w_1(x)$
4. hence load along y direction is zero and deflection in y direction is given by $w_1(x, y)$
5. This method used single trigonometric series

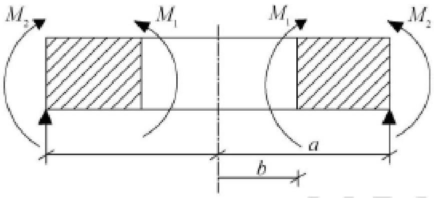
Q.3

a. Determine the transverse deflection w for a simply supported circular plate with a hole of radius a that is subject to evenly distributed moments M1 and M2 along its inner and outer borders. Therefore, locate expressions for Mr to solve up to:

$$\therefore \quad M_r = \left\{ \left[\frac{a^2 b^2 (M_1 - M_2)}{r^2\left[(a^2 - b^2)\right]} \right] - \left[\frac{b^2 M_1 - a^2 M_2}{(a^2 - b^2)} \right] \right\}$$

Answer:

4.6.1 Bending of a plate by moments M_1 and M_2 uniformly distributed along inner and outer boundaries:



Consider bending of a circular plate by moments M_1 and M_2 uniformly distributed along inner and outer boundaries. Since plate is subjected to pure bending moments the shearing force vanishes i.e. Q = 0 in the differential equation (4.6) of deflection.

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = \frac{Q}{D}$$

$$\therefore \quad \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = 0 \qquad (4.88)$$

Integrate both sides of above equation with respect to r, we get

$$\therefore \quad \left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = C_1$$

Multiply by r to the both sides of above equation and making integration with respect to r again

$$\therefore \quad r\frac{dw}{dr} = C_1\frac{r^2}{2} + C_2$$

Divide by r to the both sides and integrate with respect to r

$$\therefore \quad w = C_1\frac{r^2}{4} + C_2\log r + C_3 \qquad (4.89)$$

This is also written as

$$\therefore \quad w = C_1 \frac{r^2}{4} + C_2 \log\left(\frac{r}{a}\right) + C_3 \qquad (4.90)$$

The constants of integration are now to be determined from the conditions at the edges. Since plate is simply supported along the outer edge, we have

$$\begin{aligned} w &= 0 && \text{at } r = a \\ M_r &= M_2 && \text{at } r = a \\ M_r &= M_1 && \text{at } r = b \end{aligned} \qquad (4.91)$$

Since deflection is zero at the supporting edges, using above boundary condition put $r = a$ in the equation of deflection (4.90) and equate with zero, we get

$$\therefore \quad w|_{r=a} = C_1 \frac{a^2}{4} + C_2 \log\left(\frac{a}{a}\right) + C_3$$

$$\therefore \quad C_1 \frac{a^2}{4} + C_3 = 0 \qquad (4.91\text{ a})$$

Differentiate equation (4.90) twice with respect to r to determine bending moments, hence

$$\therefore \quad \frac{dw}{dr} = C_1 \frac{r}{2} + \frac{C_2}{r} \qquad (4.92)$$

$$\therefore \quad \frac{d^2w}{dr^2} = \frac{C_1}{2} - \frac{C_2}{r^2} \qquad (4.93)$$

Therefore put equation (4.92) and (4.93) in equation (4.2) to find bending moment M_r

$$\therefore \quad M_r = -D\left[\frac{C_1}{2} - \frac{C_2}{r^2} + \frac{\upsilon}{r}\left(C_1 \frac{r}{2} + \frac{C_2}{r}\right)\right]$$

$$\therefore \quad M_r = -D\left[(1+\upsilon)\frac{C_1}{2} - \frac{C_2}{r^2}(1-\upsilon)\right] \qquad (4.94)$$

This moment must be equal to M_1 for $r = b$ and equal to M_2 for $r = a$, hence equation (4.94) becomes

$$\therefore \quad M_r\big|_{r=b} = -D\left[(1+\upsilon)\frac{C_1}{2} - \frac{C_2}{b^2}(1-\upsilon)\right]$$

$$\therefore \quad M_1 = -D\left[(1+\upsilon)\frac{C_1}{2} - \frac{C_2}{b^2}(1-\upsilon)\right] \qquad (4.95)$$

$$\therefore \quad M_r\big|_{r=a} = -D\left[(1+\upsilon)\frac{C_1}{2} - \frac{C_2}{a^2}(1-\upsilon)\right]$$

$$\therefore \quad M_2 = -D\left[(1+\upsilon)\frac{C_1}{2} - \frac{C_2}{a^2}(1-\upsilon)\right] \qquad (4.96)$$

Subtract equation (4.96) from equation (4.95) we get

$$\therefore \quad M_1 - M_2 = -D\left[(1+\upsilon)\frac{C_1}{2} - \frac{C_2}{b^2}(1-\upsilon)\right] + D\left[(1+\upsilon)\frac{C_1}{2} - \frac{C_2}{a^2}(1-\upsilon)\right]$$

$$\therefore \quad M_1 - M_2 = DC_2(1-\upsilon)\left[\frac{1}{b^2} - \frac{1}{a^2}\right]$$

$$\therefore \quad \frac{(M_1 - M_2)}{(1-\upsilon)D\left[\frac{1}{b^2} - \frac{1}{a^2}\right]} = C_2$$

$$\therefore \quad C_2 = \frac{a^2 b^2 (M_1 - M_2)}{(1-\upsilon)D\left[(a^2 - b^2)\right]} \qquad (4.97)$$

Put in the equation (4.95) to find out C_1

$$\therefore \quad M_1 = -D\left[(1+\upsilon)\frac{C_1}{2} - \frac{1}{b^2}\frac{a^2 b^2 (M_1 - M_2)}{(1-\upsilon)D\left[(a^2 - b^2)\right]}(1-\upsilon)\right]$$

$$\therefore \quad M_1 = -D\left[(1+\upsilon)\frac{C_1}{2} - \frac{a^2(M_1 - M_2)}{D\left[(a^2 - b^2)\right]}\right]$$

$$\therefore \quad M_1 - \frac{a^2(M_1 - M_2)}{\left[(a^2 - b^2)\right]} = -D(1+\upsilon)\frac{C_1}{2}$$

	$\therefore \quad -C_1 = \frac{2}{D(1+\upsilon)}\left[\frac{M_1(a^2-b^2)-a^2(M_1-M_2)}{(a^2-b^2)}\right]$ $\therefore \quad -C_1 = \frac{2}{D(1+\upsilon)}\left[\frac{a^2M_2-b^2M_1}{(a^2-b^2)}\right]$ $\therefore \quad C_1 = \frac{2}{D(1+\upsilon)}\left[\frac{b^2M_1-a^2M_2}{(a^2-b^2)}\right] \quad (4.98)$ To determine the constant C_3 in equation (4.90), the deflection at the edges of plate must be considered. Therefore put integration constant C_1 from equation (4.98) in the equation (4.91 a). $\therefore \quad C_3 = -C_1\frac{a^2}{4}$ $\therefore \quad C_3 = \frac{a^2}{2D(1+\upsilon)}\left[\frac{a^2M_2-b^2M_1}{(a^2-b^2)}\right] \quad (4.99)$ Substitute equations (4.97), (4.98) and (4.99) in the equation (4.90) to obtain the expression for deflection of the plate. $\therefore \quad w = \left\{\begin{array}{l}\frac{2}{D(1+\upsilon)}\frac{r^2}{4}\left[\frac{b^2M_1-a^2M_2}{(a^2-b^2)}\right]+\frac{a^2b^2(M_1-M_2)}{(1-\upsilon)D\left[(a^2-b^2)\right]}\log\left(\frac{r}{a}\right)\\+\frac{a^2}{2D(1+\upsilon)}\left[\frac{a^2M_2-b^2M_1}{(a^2-b^2)}\right]\end{array}\right\}$ $\therefore \quad w = \left\{+\frac{a^2b^2(M_1-M_2)}{(1-\upsilon)D\left[(a^2-b^2)\right]}\log\left(\frac{r}{a}\right)+\frac{(a^2-r^2)}{2D(1+\upsilon)}\left[\frac{a^2M_2-b^2M_1}{(a^2-b^2)}\right]\right\} \quad (4.100)$ Substitute integration constant C_1 and C_2 from equation (4.97) and (4.98) in the equation (4.94) for bending moments of the plate. $\therefore \quad M_r = -D\left[\left\{(1+\upsilon)\frac{1}{2}\frac{2}{D(1+\upsilon)}\left[\frac{b^2M_1-a^2M_2}{(a^2-b^2)}\right]\right\}-\left\{\frac{a^2b^2(M_1-M_2)}{(1-\upsilon)D\left[(a^2-b^2)\right]}\frac{(1-\upsilon)}{r^2}\right\}\right]$ $\therefore \quad M_r = \left\{\left[\frac{a^2b^2(M_1-M_2)}{r^2\left[(a^2-b^2)\right]}\right]-\left[\frac{b^2M_1-a^2M_2}{(a^2-b^2)}\right]\right\} \quad (4.101)$
	b. Construct the governing differential equation for a circular plate under axisymmetric loading starting from first principles. (Deflection Equation for Uniformly Loaded Circular Plate)

4.2 Equation of Deflection for Uniformly Loaded Circular Plate: If a circular plate of radius a carries a load of intensity q uniformly distributed over the entire surface of the plate,

Now multiply equation (4.8) by r

$$\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right]=\frac{qr}{D}$$

Integrate w.r.t r

$$\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right]=\frac{q}{D}\frac{r^2}{2}+C_1$$

where C_1 is a constant of integration to be found later from the conditions at the center and at the edge of the plate. Dividing both sides of above equation by r and making second integration, we find

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)=\frac{q}{D}\frac{r^2}{4}+C_1\log r+C_2$$

Multiply by r to the both sides of the above equation and performing integration with respect to r, we obtained

$$r\frac{dw}{dr}=\frac{q}{D}\frac{r^4}{16}+C_1\left[\frac{r^2}{2}\log r-\frac{r^2}{4}\right]+C_2\frac{r^2}{2}+C_3$$

Divide by r to the both sides of the equation, we get

$$\frac{dw}{dr}=\frac{q}{D}\frac{r^3}{16}+C_1\left[\frac{r}{2}\log r-\frac{r}{4}\right]+C_2\frac{r}{2}+\frac{C_3}{r}$$

Integrate w.r.t. r

$$w=\frac{q}{D}\frac{r^4}{64}+C_1\left[\frac{r^2}{4}\log r-\frac{r^2}{4}\right]+C_2\frac{r^2}{4}+C_3\log r+C_4 \qquad (4.9)$$

Rearranging the term we get

$$w=C_1\log r+C_2\log r.r^2+C_3r^2+C_4+\frac{q}{D}\frac{r^4}{64} \qquad (4.10)$$

The above equation gives the deflection of plate subjected to uniformly distributed load.

Let us now calculate the constants of integration for various particular cases.

4.2.1 Simply supported circular plate subjected to uniformly distributed load:

From generalized expression (4.10) for deflection of deflected surface of circular plate subjected to uniformly distributed load.

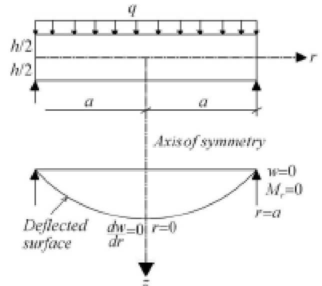
$$w=C_1\log r+C_2\log r.r^2+C_3r^2+C_4+\frac{q}{D}\frac{r^4}{64}$$

c. Determine the transverse deflection w for a circular plate with radius a that is simply supported and under evenly distributed load q. Therefore, determine the formulas for Mr (Simply supported circular plate exposed to uniformly distributed load).

$$M_r=-D\left[2C_3+\frac{q}{D}\frac{3r^2}{16}+\frac{\upsilon}{r}\left(2C_3r+\frac{q}{D}\frac{r^3}{16}\right)\right]$$ **and** $$\therefore \quad w=\frac{qa^4}{64D}\frac{(5+\upsilon)}{(1+\upsilon)}$$

From generalized expression (4.10) for deflection of deflected surface of circular plate subjected to uniformly distributed load.

$$w = C_1 \log r + C_2 \log r . r^2 + C_3 r^2 + C_4 + \frac{q}{D}\frac{r^4}{64}$$



Note: since the deflection, moment and transverse shear are to be finite at the centre of plate $(r = 0)$. Therefore the constants C_1 and C_2 have to remain zero.

Therefore above equation becomes

$$\therefore \quad w = C_3 r^2 + C_4 + \frac{q}{D}\frac{r^4}{64} \qquad (4.11)$$

The constants of integration are now to be determined from the conditions at the edges of the plate.

$$w = 0 \qquad \text{for } r = a \qquad (4.12a)$$

$$M_r = 0 \qquad \text{for } r = a \qquad (4.12b)$$

Using boundary condition (4.12a), substitute $r = a$ in the equation (4.11) and equate with zero, we obtain

$$\therefore \quad C_3 a^2 + C_4 + \frac{q}{D}\frac{a^4}{64} = 0 \qquad (4.13)$$

Differentiate equation (4.11) with respect to r upto second order to find the moment of the plate

$$\therefore \quad \frac{dw}{dr} = 2C_3 r + \frac{q}{D}\frac{4r^3}{64} = 2C_3 r + \frac{q}{D}\frac{r^3}{16} \qquad (4.14)$$

$$\therefore \quad \frac{d^2w}{dr^2} = 2C_3 + \frac{q}{D}\frac{3r^2}{16} \qquad (4.15)$$

Put equation (4.14) and (4.15) into the equation (4.2)

$$M_r = -D\left[2C_3 + \frac{q}{D}\frac{3r^2}{16} + \frac{\upsilon}{r}\left(2C_3 r + \frac{q}{D}\frac{r^3}{16}\right)\right] \qquad (4.15\text{ a})$$

Since this bending moment will vanish at the edges *i.e.* at $r = a$, Use equation (4.12b) to obtained constant C_3 by equating equation (4.15 a) with zero

$$\therefore \quad -D\left[2C_3 + \frac{q}{D}\frac{3a^2}{16} + \frac{\upsilon}{a}\left(2C_3 a + \frac{q}{D}\frac{a^3}{16}\right)\right] = 0$$

$$\therefore \quad 2C_3 + 2\upsilon C_3 + \frac{q}{D}\frac{3a^2}{16} + \frac{q}{D}\frac{a^2}{16} = 0$$

$$\therefore \quad 2C_3(1+\upsilon) + \frac{q}{D}\frac{a^2}{16}(3+\upsilon) = 0$$

$$\therefore \quad C_3 = -\frac{q}{D}\frac{a^2}{32}\frac{(3+\upsilon)}{(1+\upsilon)} \qquad (4.16)$$

Put constant C_3 from equation (4.16) into the equation (4.13) to obtain constant C_4

$$\therefore \quad -\frac{q}{D}\frac{a^2}{32}\frac{(3+\upsilon)}{(1+\upsilon)} a^2 + C_4 + \frac{q}{D}\frac{a^4}{64} = 0$$

$$\therefore \quad C_4 = \frac{q}{D}\frac{a^4}{32}\frac{(3+\upsilon)}{(1+\upsilon)} - \frac{q}{D}\frac{a^4}{64}$$

$$\therefore \quad C_4 = \frac{q}{D}\frac{a^4}{64}\left(\frac{2(3+\upsilon)}{(1+\upsilon)} - 1\right)$$

$$\therefore \quad C_4 = \frac{q}{D}\frac{a^4}{64}\left(\frac{6+2\upsilon-1-\upsilon}{(1+\upsilon)}\right)$$

$$\therefore \quad C_4 = \frac{q}{D}\frac{a^4}{64}\frac{(5+\upsilon)}{(1+\upsilon)} \tag{4.17}$$

Substitute constants C_3 and C_4 from equations (4.16) and (4.17) into the equation (4.11) to obtain deflection of the plate, hence

$$\therefore \quad w = -\frac{q}{D}\frac{a^2}{32}\frac{(3+\upsilon)}{(1+\upsilon)}r^2 + \frac{q}{D}\frac{a^4}{64}\frac{(5+\upsilon)}{(1+\upsilon)} + \frac{q}{D}\frac{r^4}{64}$$

$$\therefore \quad w = \frac{q}{64D}\left(-\frac{2(3+\upsilon)}{(1+\upsilon)}a^2r^2 + \frac{(5+\upsilon)}{(1+\upsilon)}a^4 + r^4\right)$$

$$\therefore \quad w = \frac{qa^4}{64D}\left(-\frac{2(3+\upsilon)}{(1+\upsilon)}\frac{r^2}{a^2} + \frac{(5+\upsilon)}{(1+\upsilon)} + \frac{r^4}{a^4}\right)$$

$$\therefore \quad w = \frac{qa^4}{64D}\left[\left(\frac{r}{a}\right)^4 - \frac{2(3+\upsilon)}{(1+\upsilon)}\left(\frac{r}{a}\right)^2 + \frac{(5+\upsilon)}{(1+\upsilon)}\right] \tag{4.18}$$

Deflection is maximum at the center of the plate *i.e.* at $r = 0$

$$\therefore \quad w = \frac{qa^4}{64D}\frac{(5+\upsilon)}{(1+\upsilon)} \tag{4.19}$$

d. Derive the formula for bending a circular plate symmetrically under simply supported conditions.

4.1 Differential Equation for Symmetrical bending of circular plates:

Consider Symmetrical bending of circular plates shown in figure. Let us take origin of coordinates '*O*' at the centre of deflected plate as shown in figure. Let *r* denoted the radial distances of points in the middle plane of the plate and *w* be the deflection of the plate in *z* direction at any point A.

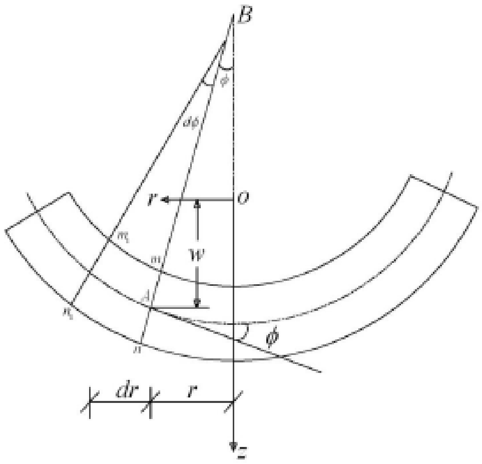
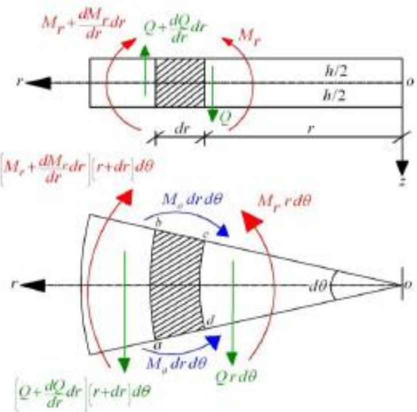


Figure: Bending of circular plate

$$\left(M_r + \frac{dM_r}{dr} dr\right)(r+dr)\,d\theta$$

The couples on the sides' *ad* and *bc* of the element are each $M_\theta\, dr$ and they give a resultant couple in the plane *roz* $\quad M_\theta\, dr\, d\theta$



Summing up the moments with proper sign and neglecting the moment due to external load on the element as a small quantity of higher order, we obtain the following equation of equilibrium of the element *abcd*.

$$\therefore \quad \left(M_r + \frac{dM_r}{dr} dr\right)(r+dr)\,d\theta - M_r\, r\, d\theta - M_\theta\, dr\, d\theta + Q\, r\, d\theta\, dr = 0$$

$$\therefore \quad M_r (r+dr)\,d\theta + \frac{dM_r}{dr} dr\,(r+dr)\,d\theta - M_r\, r\, d\theta - M_\theta\, dr\, d\theta + Q\, r\, d\theta\, dr = 0$$

$$\therefore \quad M_r\, r\, d\theta + M_r\, dr\, d\theta + r\frac{dM_r}{dr} dr\, d\theta + \frac{dM_r}{dr} dr\, dr\, d\theta - M_r\, r\, d\theta - M_\theta\, dr\, d\theta + Q\, r\, d\theta\, dr = 0$$

Since dr and $d\theta$ are very small its higher power is neglected.

$$\therefore \quad M_r\, dr\, d\theta + r\frac{dM_r}{dr} dr\, d\theta - M_\theta\, dr\, d\theta + Q\, r\, d\theta\, dr = 0$$

$$\therefore \quad M_r + r\frac{dM_r}{dr} - M_\theta + Qr = 0 \qquad \text{(a)}$$

Put value of M_r and M_θ from equation (4.2) and (4.3) in (a)

$$\therefore \quad D\left[\frac{d\phi}{dr} + \frac{\upsilon}{r}\phi\right] + \frac{d}{dr}\, rD\left[\frac{d\phi}{dr} + \frac{\upsilon}{r}\phi\right] - D\left[\upsilon\frac{d\phi}{dr} + \frac{\phi}{r}\right] + Qr = 0$$

$$\therefore \quad \frac{d\phi}{dr} + \frac{\upsilon}{r}\phi + r\frac{d^2\phi}{dr^2} + \left(r\frac{\upsilon}{r}\frac{d\phi}{dr} + r\upsilon\phi\left[-\frac{1}{r^2}\right]\right) - \upsilon\frac{d\phi}{dr} - \frac{\phi}{r} = -\frac{Qr}{D}$$

$$\therefore \quad \frac{d\phi}{dr} + \frac{\upsilon}{r}\phi + r\frac{d^2\phi}{dr^2} + \upsilon\frac{d\phi}{dr} - \frac{\upsilon\phi}{r} - \upsilon\frac{d\phi}{dr} - \frac{\phi}{r} = -\frac{Qr}{D}$$

$$\therefore \quad \frac{1}{r}\frac{d\phi}{dr} + \frac{d^2\phi}{dr^2} - \frac{\phi}{r^2} = -\frac{Q}{D} \qquad (4.4)$$

Put $\phi = -\dfrac{dw}{dr}$

$$\therefore \quad \frac{1}{r}\frac{d}{dr}\left[-\frac{dw}{dr}\right] + \frac{d^2}{dr^2}\left[-\frac{dw}{dr}\right] - \frac{1}{r^2}\left[-\frac{dw}{dr}\right] = -\frac{Q}{D}$$

$$\therefore \quad -\frac{d^3w}{dr^3} - \frac{1}{r}\frac{d^2w}{dr^2} + \frac{1}{r^2}\frac{dw}{dr} = -\frac{Q}{D}$$

$$\therefore \quad \frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} = \frac{Q}{D} \qquad (4.5)$$

In any particular case of symmetrically loaded circular plate the shearing force Q can easily being calculated by dividing the load distributed the circle of radius r by $2\pi r$; then equation (4.4) or (4.5) can be used to determine the slope ϕ and the deflection w of the plate. The integration of these equations is simplified if we observe that they can be put in the following form.

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = \frac{Q}{D} \qquad (4.6)$$

If Q is represented by a function r, this equation can be integrated without any difficulty in each particular case. Sometimes it is advantageous to represent the right hand side of equation (4.6) as a function of intensity q of the load distributed over the plate.

For this purpose we multiply both sides of the equation by $2\pi r$. Then, observing that

$$\therefore \quad 2\pi r Q = \int_0^r 2\pi r q\, dr$$

$$\therefore \quad Q = \int_0^r q\, dr$$

Put in the equation (4.6), and multiply both sides of the equation by r, we obtained

$$\therefore \quad r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = \frac{1}{D}\int_0^r q\, r\, dr$$

Differentiate the term inside the bracket with respect to r, we get

$$\therefore \quad r\frac{d}{dr}\left[\frac{1}{r}\left(r\frac{d^2w}{dr^2} + \frac{dw}{dr}\right)\right] = \frac{1}{D}\int_0^r q\, r\, dr$$

Again differentiate left hand side of the above equation with respect to r, we obtained

$$\therefore \quad r\left[\frac{1}{r}\left(\left\{r\frac{d^3w}{dr^3} + \frac{d^2w}{dr^2}\right\} + \frac{d^2w}{dr^2}\right) - \frac{1}{r^2}\left(r\frac{d^2w}{dr^2} + \frac{dw}{dr}\right)\right] = \frac{q}{D}\int_0^r r\, dr$$

$$\therefore \quad r\frac{d^3w}{dr^3} + 2\frac{d^2w}{dr^2} - \frac{d^2w}{dr^2} - \frac{1}{r}\frac{dw}{dr} = \frac{q}{D}\int_0^r r\, dr$$

Differentiate both sides of the equation w.r.t. r to eliminate integral term

$$\therefore \quad \left(r\frac{d^4w}{dr^4} + \frac{d^3w}{dr^3}\right) + \frac{d^3w}{dr^3} - \left(\frac{1}{r}\frac{d^2w}{dr^2} + \frac{dw}{dr}\left[-\frac{1}{r^2}\right]\right) = \frac{q\,r}{D}$$

$$\therefore \quad \frac{d^4w}{dr^4} + \frac{2}{r}\frac{d^3w}{dr^3} - \frac{1}{r}\frac{d^2w}{dr^2} + \frac{1}{r^3}\frac{dw}{dr} = \frac{q}{D} \qquad (4.7)$$

This can also be written as

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right] = \frac{q}{D} \qquad (4.8)$$

Q.4

a) Describe the benefits and drawbacks of shell constructions in comparison to plates and list the presumptions and their implications in the general theory of thin elastic shells.

Advantages of shell structures:

1. Major load is carried through membrane action and not through bending.

2. Due to this small thickness can be used, requires less material and is economical.

3. Shapes are architecturally beautiful and streamlined.

4. Large floor area uninterrupted by supports is obtained with shells.

Disadvantage of shell structures:

1. Difficult to analyze.

2. Difficult to construct due to complex geometry.

3. Cannot be used as a floor.

Assumptions made in theory of thin elastic shell:

1. Stresses in z direction is neglected in comparison with and σ_x and σ_y

2. Straight line normal to undeformed middle surface remains straight and normal to deformed middle surface.

3. Displacements are small enough so that the changes in geometry of shell are negligible for equilibrium.

4. The material is linearly elastic, homogenous.

Due to assumption (1) (called Love"s Hypothesis) the 3-D problems are converted into 2-D problems. In assumption (2) effect of shear deformation is neglected. Assumption (3) makes equations simple and (4) avoid orthotropiy, non-linearity and discontinuities.

b) Sort shells according to their geometry. Give examples to illustrate each type.

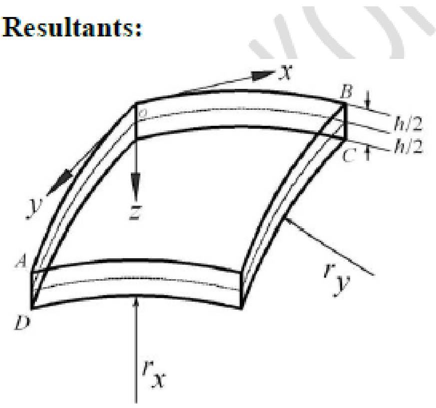
Answer:

Shells can be classified based on their geometry as follows:

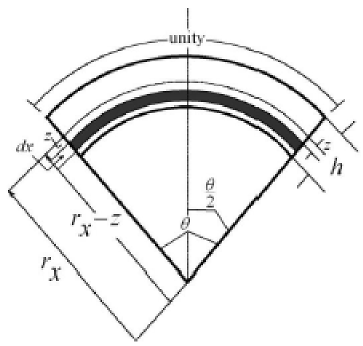
1. Singly Curved Shells: Curved in one direction only (e.g., cylindrical shells).
2. Doubly Curved Shells: Curved in two directions.
3. Synclastic: Curvatures in the same direction (e.g., spherical and ellipsoidal shells).
4. Anticlastic: Curvatures in opposite directions (e.g., hyperbolic paraboloid).
5. Developable Shells: Can be developed onto a plane without stretching (e.g., conical shells).
6. Non-developable Shells: Cannot be flattened without distortion (e.g., spherical caps).

c) Draw a cylindrical shell. On the shell surface, display the stress results for a small element. Determine the equilibrium equations for this component.

5.7 Determination of Stress Resultants:



Consider an infinitesimal small element of shell cut out by sections parallel to x and y axes and normal to mid plane. Let x and y be the directions of principal curvatures and r_x, r_y be the radii of principal curvatures for the element. Let the stresses $\sigma_x, \sigma_y, \tau_{xy} = \tau_{yx}, \gamma_{xz}$ and γ_{yz} be acting at a point in the material at distance z from the mid surface as shown in figure.



Using similarities of triangle

$$\frac{1}{r_x} = \frac{L_x}{r_x - z} \quad \Rightarrow \quad L_x = \frac{r_x - z}{r_x}$$

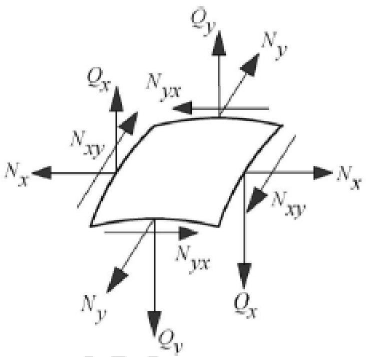
$$\therefore \quad L_x = 1 - \frac{z}{r_x}$$

Similarly $L_y = 1 - \frac{z}{r_y}$

Hence along x direction length of arc will be

$\left(1 - \frac{z}{r_x}\right)$ and along y direction $\left(1 - \frac{z}{r_y}\right)$

Therefore stress resultants per unit length of edge of shell become



$$\left.\begin{aligned}
N_x &= \int_{-h/2}^{+h/2} \sigma_x \left(1 - \frac{z}{r_y}\right) dz \\
N_y &= \int_{-h/2}^{+h/2} \sigma_y \left(1 - \frac{z}{r_x}\right) dz \\
N_{xy} &= \int_{-h/2}^{+h/2} \tau_{xy} \left(1 - \frac{z}{r_y}\right) dz \\
N_{yx} &= \int_{-h/2}^{+h/2} \tau_{xy} \left(1 - \frac{z}{r_x}\right) dz
\end{aligned}\right\}$$

$$\left.\begin{aligned}
Q_x &= \int_{-h/2}^{+h/2} \tau_{xz} \left(1 - \frac{z}{r_y}\right) dz & \qquad Q_y &= \int_{-h/2}^{+h/2} \tau_{yz} \left(1 - \frac{z}{r_y}\right) dz \\
M_x &= \int_{-h/2}^{+h/2} \sigma_x \, z \left(1 - \frac{z}{r_y}\right) dz & \qquad M_y &= \int_{-h/2}^{+h/2} \sigma_y \, z \left(1 - \frac{z}{r_x}\right) dz \\
M_{xy} &= \int_{-h/2}^{+h/2} \tau_{xy} \, z \left(1 - \frac{z}{r_y}\right) dz & \qquad M_{yx} &= \int_{-h/2}^{+h/2} \tau_{xy} \, z \left(1 - \frac{z}{r_x}\right) dz
\end{aligned}\right\} \quad (5.1)$$

d) Describe the membrane hypothesis of shells of revolution. What are its presumptions and constraints?

Answer:

Membrane theory assumes that the shell carries only in-plane forces (no bending moments). Applicable to shells of revolution like spheres, cones, and cylinders.

Assumptions:

- Shell is thin.
- No bending or twisting moments.
- Deformations are small.
- Loads are axisymmetric.

Limitations:

- Not valid near boundaries or under concentrated loads.
- Cannot capture bending effects or edge disturbances.

Applications: Domes, silos, pressure vessels under symmetric loading.

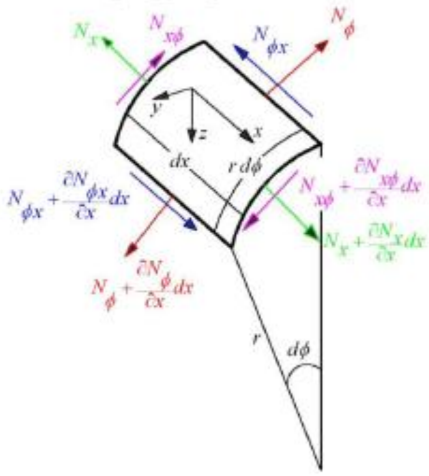
Q.5

a. Describe the membrane theory of cylindrical shells.

5.8 Membrane theory of cylindrical shell:

Let consider, an element is cut from the shell by two adjacent generators and two cross sections perpendicular to the x axis, and its position is defined by the coordinates x and the angle ϕ. In addition a load will be distributed over the surface of the element, the components of the intensity of this load being denoted, by X, Y and Z.

Let x axis be taken along the length, y tangent to the cross-section and z is along normal to the surface. The stress resultants acting on edges of a small element having size $dx.rd\phi$ are as shown in figure. Considering the equilibrium of the element and summing up the forces in x direction we obtained



$$\therefore \quad -N_x\, r\,d\phi + \left[N_x + \frac{\partial N_x}{\partial x}dx\right] r\,d\phi - N_{\phi x}\,dx + \left[N_{\phi x} + \frac{\partial N_{\phi x}}{\partial \phi}d\phi\right]dx + X\,r\,d\phi\,dx = 0$$

$$\therefore \quad \frac{\partial N_x}{\partial x}dx\,r\,d\phi + \frac{\partial N_{\phi x}}{\partial \phi}d\phi\,dx + X\,r\,d\phi\,dx = 0$$

$$\therefore \quad \frac{\partial N_x}{\partial x} + \frac{1}{r}\frac{\partial N_{\phi x}}{\partial \phi} + X = 0 \qquad (5.29)$$

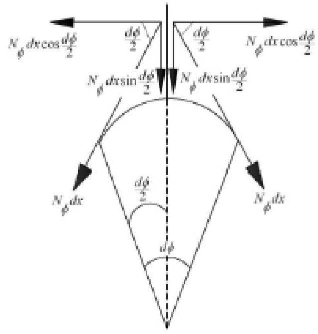
Similarly, the force in the direction of the tangent to the normal cross section, i.e. in the y direction gives the corresponding equation of equilibrium, i.e. $\Sigma F_y = 0$

$$\therefore \quad -N_\phi\,dx + \left[N_\phi + \frac{\partial N_\phi}{\partial \phi}d\phi\right]dx - N_{x\phi}\,r\,d\phi + \left[N_{x\phi} + \frac{\partial N_{x\phi}}{\partial x}dx\right] r\,d\phi + Y\,r\,d\phi\,dx = 0$$

$$\therefore \quad \frac{\partial N_\phi}{\partial \phi}d\phi\,dx + \frac{\partial N_{x\phi}}{\partial x}r\,d\phi\,dx + Y\,r\,d\phi\,dx = 0$$

$$\therefore \quad \frac{1}{r}\frac{\partial N_\phi}{\partial \phi} + \frac{\partial N_{x\phi}}{\partial x} + Y = 0 \qquad (5.30)$$

The forces acting in the direction of the normal to the shell, i.e. in the z direction



Total downward force in z direction $= N_\phi\,dx\,\sin\left(\frac{d\phi}{2}\right) + N_\phi\,dx\,\sin\left(\frac{d\phi}{2}\right)$

$$= 2N_\phi\,dx\left(\frac{d\phi}{2}\right) = N_\phi\,dx\,d\phi$$

$$\therefore \quad N_\phi\,dx\,d\phi + Z\,r\,d\phi\,dx = 0$$

$$\therefore \quad N_\phi + Z\,r = 0$$

$$\therefore \quad N_\phi = -Z\,r \qquad (5.31)$$

Note: We find N_ϕ from equation (5.31) *and $N_{x\phi}$, N_x by integration of equation* (5.30) *and* (5.29)

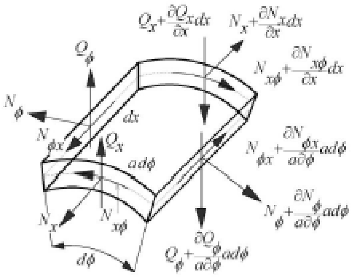
b. **Write in brief about the D-K-J theory for cylindrical shells while taking the Bending Theory: Equilibrium equation into consideration.**

6.4 The D-K-J Theory (Donnell – Karman – Jenkins Theory)

The simplest among the so-called exact theories which takes into account M_x, $M_{x\phi}$ and Q_x ignored in the finsterwalder theory is the D-K-J theory in which the three displacements u, v and w appear in uncoupled form. The theory appears to be due to Donnell, who first used it in connection with his studies on the stability of thin-walled circular cylinders in 1933-1934. Karman and Tsien also employed the same theory in 1941 in their investigation on the buckling of cylindrical shells. Its presentation in a form suitable for the analysis of cylindrical shell roofs appeared in a book by Jenkins published in 1947. The theory is appropriately called as Donnell – Karman – Jenkins theory.

6.4.1 Equations of Equilibrium:

The equilibrium equations already derived in finsterwalder theory will now be modified to take into account M_x, $M_{x\phi}$ and Q_x. It will be assumed that $M_{x\phi} = M_{\phi x}$ and $N_{x\phi} = N_{\phi x}$



Referring to above figure, the following equations of equilibrium may be set up. Equating all forces in the x direction to zero we get

$$\Sigma F_x = 0$$

$$\therefore \quad -N_x\, a\, d\phi + \left(N_x + \frac{\partial N_x}{\partial x} dx \right) a\, d\phi - N_{\phi x} dx + \left(N_{\phi x} + \frac{\partial N_{\phi x}}{a\, \partial \phi} a\, d\phi \right) dx = 0$$

$$\therefore \quad \frac{\partial N_x}{\partial x} dx\, a\, d\phi + \frac{\partial N_{\phi x}}{a\, \partial \phi} a\, d\phi\, dx = 0$$

$$\therefore \quad \frac{\partial N_x}{\partial x} + \frac{1}{a} \frac{\partial N_{\phi x}}{\partial \phi} = 0 \qquad (6.29)$$

Equating the sum of all the forces acting in ϕ direction to zero, we get

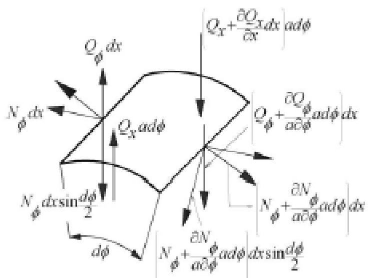
$$\Sigma F_\phi = 0$$

$$\therefore \quad -N_\phi\, dx + \left(N_\phi + \frac{\partial N_\phi}{a\, \partial \phi} a\, d\phi \right) dx - N_{x\phi}\, a\, d\phi + \left(N_{x\phi} + \frac{\partial N_{x\phi}}{\partial x} dx \right) a\, d\phi = 0$$

$$\therefore \quad \frac{\partial N_\phi}{a\, \partial \phi} a\, d\phi\, dx + \frac{\partial N_{x\phi}}{\partial x} dx\, a\, d\phi = 0$$

$$\therefore \quad \frac{\partial N_\phi}{\partial \phi} + a \frac{\partial N_{x\phi}}{\partial x} = 0 \qquad (6.30)$$

The third equation of equilibrium is derived by equating all forces in the direction of the inward normal drawn at the midpoint of the shell element to zero.



$$N_\phi\, dx \sin\frac{d\phi}{2} + \left(N_\phi + \frac{\partial N_\phi}{a\, \partial \phi} a\, d\phi \right) dx \sin\frac{d\phi}{2} - Q_\phi\, dx + \left(Q_\phi + \frac{\partial Q_\phi}{a\, \partial \phi} a\, d\phi \right) dx$$

$$-Q_x\, a\, d\phi + \left(Q_x + \frac{\partial Q_x}{\partial x} dx \right) a\, d\phi = 0$$

Neglecting higher powers of dx and $d\phi$ we obtained

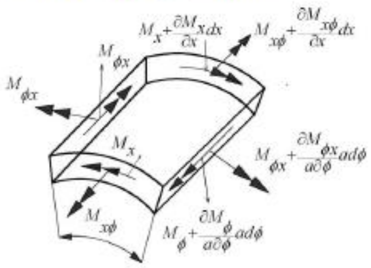
$$\therefore \quad 2N_\phi\, dx \sin\frac{d\phi}{2} + \frac{\partial Q_\phi}{\partial \phi} d\phi\, dx + \frac{\partial Q_x}{\partial x} dx\, a\, d\phi = 0$$

$$\therefore \quad N_\phi\, dx\, d\phi + \frac{\partial Q_\phi}{\partial \phi} d\phi\, dx + \frac{\partial Q_x}{\partial x} dx\, a\, d\phi = 0$$

On simplifying, we get

$$\therefore \quad N_\phi + \frac{\partial Q_\phi}{\partial \phi} + \frac{\partial Q_x}{\partial x} a = 0 \qquad (6.31)$$

Another equation of equilibrium results from equating the sum of moments of all forces about the generatrix. Moments of all forces about the generatrix *AD*.



$$\therefore \quad \Sigma M_{AD} = 0$$

$$\therefore -M_\phi\,dx + \left(M_\phi + \frac{\partial M_\phi}{a\,\partial\phi}\,a\,d\phi\right)dx - M_{x\phi}\,a\,d\phi + \left(M_{x\phi} + \frac{\partial M_{x\phi}}{\partial x}dx\right)a\,d\phi$$

$$-\left(Q_\phi + \frac{\partial Q_\phi}{a\,\partial\phi}a\,d\phi\right)dx\,a\,d\phi = 0$$

Neglecting higher powers of dx and $d\phi$ we obtained

$$\therefore \quad \frac{\partial M_\phi}{\partial\phi}\,dx\,d\phi + a\frac{\partial M_{x\phi}}{\partial x}\,dx\,d\phi - a\,Q_\phi\,dx\,d\phi = 0$$

$$\therefore \quad \frac{\partial M_\phi}{\partial\phi} + a\frac{\partial M_{x\phi}}{\partial x} - a\,Q_\phi = 0 \qquad (6.32)$$

Moments of all forces about the generatrix AB.

$$\therefore \quad \Sigma M_{AB} = 0$$

$$\therefore \quad -M_x\,a\,d\phi + \left(M_x + \frac{\partial M_x}{\partial x}dx\right)a\,d\phi - M_{\phi x}\,dx$$

$$+\left(M_{\phi x} + \frac{\partial M_{\phi x}}{a\partial\phi}a\,d\phi\right)dx - \left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)a\,d\phi\,dx = 0$$

$$\therefore \quad a\frac{\partial M_x}{\partial x}\,dx\,d\phi + \frac{\partial M_{\phi x}}{\partial\phi}\,dx\,d\phi - a\,Q_x\,dx\,d\phi = 0$$

$$\therefore \quad a\frac{\partial M_x}{\partial x} + \frac{\partial M_{\phi x}}{\partial\phi} - a\,Q_x = 0 \qquad (6.33)$$

c. Describe the membrane and bending theories used in shell analysis.

1. Membrane Theory:

- **Assumptions**:
  - Neglects bending stresses; considers only in-plane forces (normal and shear).
- **Applicable For**:
  - Thin shells under uniform loading with continuous support.
- **Advantages**:
  - Simple and useful for preliminary design.
- **Stress Resultants**:
  - Membrane forces (N_1, N_2, N_{12}).
- **Equilibrium**:
  - Based on geometry and in-plane equilibrium equations.

Diagram: Show a thin shell element with in-plane forces N_1, N_2.

2. Bending Theory:

- **Assumptions**:
  - Considers both in-plane and bending stresses.
- **Applicable For**:
  - Edge zones, concentrated loads, discontinuities.
- **Stress Resultants**:
  - Membrane forces + Bending moments (M_1, M_2), shear forces (Q).
- **More Accurate**:
  - Useful for full structural analysis.

Comparison Table:

Feature	Membrane Theory	Bending Theory
Complexity	Simple	Complex
Accuracy	Approximate	More accurate
Applicable Regions	Middle surface	Edge regions, discontinuities
Load Types	Uniform distributed loads	Point loads, varying loads
Stress Types	Normal and shear forces	Normal, shear, bending moments

d) Describe the benefits and drawbacks of the beam approach of cylindrical shell analysis. (The Beam Theory of Lundgren)

Lundgren's beam theory provides a simple approach of analysis for cylindrical shell of long span, supported at ends. In this approach, shells are treated simply as a beam, and bending and shear stresses for this beam is calculated.

6.7.1 Advantages:

1. It brings shell analysis within the reach of those who are not familiar with advance mathematics.
2. It is also applicable to shell having nonuniform thickness (along the directrix).
3. Line load acting on the shell also treated by this.
4. Structural action of shell is easily visualized.
5. It can handle shell strengthened by ribs in longitudinal and transverse direction (stiffening Beams).
6. Application to shell with non-circular directrices is possible.

6.7.2 Assumptions:

1. Transverse deflection of shell in its plane is neglected. This assumption is replaced by Bernoulli's assumption (plane section before deformation remains plane after deformation) if shell is subjected to pure bending.
2. $M_x = M_{xy} = Q_x = 0$ (true only for a long shell)
3. Shear strain $\gamma_{x\phi}$ caused by $N_{x\phi}$ is neglected

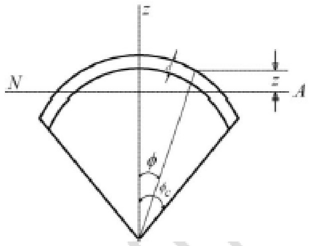
All these assumptions will be linearly valid only for long beams. It is observed that beam theory gives fairly acceptable answers for cylindrical roof shells without edge beams if L/a > 5, and shells with edge beams if L/a > 3. These limits may be used as guidelines for a designer. The shell must be uniformly loaded.

This method is carried out in two distinct steps

a) Beam analysis
b) Arch Analysis

6.7.3 Beam Analysis:

Shell regarded as a beam supported at curved edges. Bending stresses and shearing stresses are found using the beam formulae.



d = thickness of shell at any c/s

At any c/s value of stress resultants N_x is given by $N_x = \frac{M_{yy}}{I_{yy}}.Z.d$ (6.84)

M_{yy} = Bending Moment at any c/s

I_{yy} = Moment of inertia @ y-y axis

$$\therefore \quad I_{yy} = 2d \int_{\phi=0}^{\phi=\phi_c} a\,d\phi \left(a\cos\phi - a\frac{\sin\phi_c}{\phi_c} \right)^2$$

$$\therefore \quad I_{yy} = 2\,a^3\,d \int_{\phi=0}^{\phi=\phi_c} \left[\cos^2\phi - 2\cos\phi\frac{\sin\phi_c}{\phi_c} + \frac{\sin^2\phi_c}{\phi_c^2} \right] d\phi$$

$$\therefore \quad I_{yy} = 2\,a^3\,d \int_{\phi=0}^{\phi=\phi_c} \left[\frac{1+\cos2\phi}{2} - 2\,\frac{\cos\phi\sin\phi_c}{\phi_c} + \frac{\sin^2\phi_c}{\phi_c^2} \right] d\phi$$

$$\therefore \quad I_{yy} = 2\,a^3\,d \int_{\phi=0}^{\phi=\phi_c} \left[\frac{1+\cos 2\phi}{2} - 2\,\frac{\cos\phi \sin\phi_c}{\phi_c} + \frac{\sin^2\phi_c}{\phi_c^2} \right] d\phi$$

$$\therefore \quad I_{yy} = 2\,a^3\,d \left[\frac{\phi}{2} + \frac{\sin 2\phi}{4} - 2\,\frac{\sin\phi \sin\phi_c}{\phi_c} + \phi\,\frac{\sin^2\phi_c}{\phi_c^2} \right]_0^{\phi_c}$$

$$\therefore \quad I_{yy} = 2\,a^3\,d \left[\frac{\phi_c}{2} + \frac{\sin 2\phi_c}{4} - 2\,\frac{\sin^2\phi_c}{\phi_c} + \frac{\sin^2\phi_c}{\phi_c} \right]$$

$$\therefore \quad I_{yy} = a^3\,d \left[\phi_c + \frac{\sin 2\phi_c}{2} - \frac{2\sin^2\phi_c}{\phi_c} \right]$$

$$\therefore \quad I_{yy} = a^3\,d \left[\phi_c + \sin\phi_c \left(\cos\phi_c - \frac{2\sin\phi_c}{\phi_c} \right) \right] \qquad (6.85)$$

Equation (6.84) and (6.85) are applicable to only shells of symmetric cross/section.

The beam analysis also enables $N_{x\phi}$ to be found out by the use of well known $\frac{V.Q}{Ib}$ formula for the shell subjected to only vertical loading symmetrically distributed over the cross/section.

$$\therefore \quad N_{x\phi} = \frac{V.Q.d}{I_{yy} b} = \frac{V.Q.d}{I_{yy}\,2d} = \frac{V.Q}{2I_{yy}} \qquad (6.86)$$

Where, V = vertical shearing force *and* Q = moment of Area

This is given by

$$Q = a \left[\frac{\sin\phi}{\phi} - \frac{\sin\phi_c}{\phi_c} \right] 2\,a\,\phi\,d$$

$$\therefore \quad Q = 2\,a^2\,d \left[\sin\phi - \phi\,\frac{\sin\phi_c}{\phi_c} \right] \qquad (6.87)$$