K. K. Wagh Institute of Engineering Education and Research, Nashik
(An Autonomous Institute from A.Y. 2022-23)

Model Answer

End-Sem Examination-I, Winter 2025

Academic Year: 2025-2026	Semester: II
Class: F.Y. M. Tech	Program: M. Tech
Branch Code: CIV	Pattern: 2024
Name of Course: Finite Element Method	Course Code: 2404512

Q.	Answer	Marks
1	Different types of elements in FEM 1D elements…….. (1) 2D elements- CST, LST, QST, Rectangular elements, serendipity elements, Quadrilateral, 2D curve, iso-parametric, axisymmetric……. (3) 3D elements……… (2)	6
2	Explain in detail all the steps of Finite Element Method. Basic Steps in Finite Element Analysis The following steps are performed for finite element analysis. 1. Discretisation of the continuum: The continuum is divided into a number of elements by imaginary lines or surfaces. The interconnected elements may have different sizes and shapes. 2. Identification of variables: The elements are assumed to be connected at their intersecting points referred to as nodal points. At each node, unknown displacements are to be prescribed. 3. Choice of approximating functions: Displacement function is the starting point of the mathematical analysis. This represents the variation of the displacement within the element. The displacement function may be approximated in the form a linear function or a higher order function. A convenient way to express it is by polynomial expressions. The shape or geometry of the element may also be approximated. 4. Formation of the element stiffness matrix: After continuum is discretised with desired element shapes, the individual element stiffness matrix is formulated. Basically it is a minimization procedure whatever may be the approach adopted. For certain elements, the form involves a great deal of sophistication. The geometry of the element is defined in reference to the global frame. Coordinate transformation must be done for elements where it is necessary. 5. Formation of overall stiffness matrix: After the element stiffness matrices in global coordinates are formed, they are assembled to form the overall stiffness matrix. The assembly is done through the nodes which are common to adjacent elements. The overall stiffness matrix is symmetric and banded. 6. Formation of the element loading matrix: The loading forms an essential parameter in any structural engineering problem. The loading inside an element is transferred at the nodal points and consistent element matrix is formed. 7. Formation of the overall loading matrix: Like the overall stiffness matrix, the element loading matrices are assembled to form the overall loading matrix. This matrix has one column per loading case and it is either a column vector or a rectangular matrix depending on the number of loading cases. 8. Incorporation of boundary conditions: The boundary restraint conditions are to be imposed in the stiffness matrix. There are various techniques available to satisfy the boundary conditions. One is the size of the stiffness matrix may be reduced or condensed in its final form. To ease computer programming aspect and to elegantly incorporate the boundary conditions, the size of overall matrix is kept the same.	6
3	a) Using generalized co-ordinate approach find shape functions for two nodded bar element	8

Solution: Figure 5.5 shows the typical truss element. In this case nodal unknowns are displacements u_1 and u_2 along x-axis. For this element we have to select polynomial with only two constants to represent displacement at any point in the elements. Hence we select

$$u = \alpha_1 + \alpha_2 x \qquad \ldots(5.11)$$

where α_1 and α_2 are generalized coordinates. This polynomial satisfies compatibility and completeness requirement. Writing equation 5.11 in the matrix form we have,

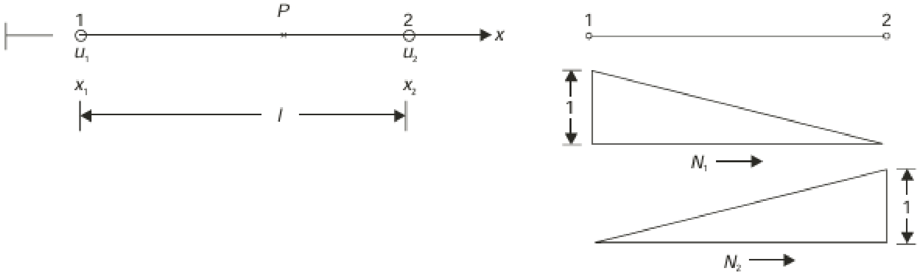


Fig. 5.5 *Bar/Truss element with two nodes*

$$u = \begin{bmatrix}1 & x\end{bmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\end{Bmatrix}$$

since $u = u_1$ at node 1 and equal to u_2 at node 2, we have

$$\{\delta\} = \begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{bmatrix}1 & x_1\\ 1 & x_2\end{bmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\end{Bmatrix}$$

$$\begin{Bmatrix}\alpha_1\\ \alpha_2\end{Bmatrix} = \begin{bmatrix}1 & x_1\\ 1 & x_2\end{bmatrix}^{-1}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \frac{1}{x_2 - x_1}\begin{bmatrix}x_2 & -1\\ -x_1 & 1\end{bmatrix}^T\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \frac{1}{l}\begin{bmatrix}x_2 & -x_1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

$$\therefore\ u = \begin{bmatrix}1 & x\end{bmatrix}\begin{Bmatrix}\alpha_1\\ \alpha_2\end{Bmatrix} = \begin{bmatrix}1 & x\end{bmatrix}\frac{1}{l}\begin{bmatrix}x_2 & -x_1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

$$= \frac{1}{l}\begin{bmatrix}x_2 - x & -x_1 + x\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{bmatrix}\dfrac{x_2 - x}{l} & \dfrac{x - x_1}{l}\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

$$= \begin{bmatrix}N_1 & N_2\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = N_1 u_1 + N_2 u_2$$

where $N_1 = \dfrac{x_2 - x}{l}$ and $N_2 = \dfrac{x - x_1}{l}$

Thus the shape function $[N]$ is

$$[N] = \begin{bmatrix}N_1 & N_2\end{bmatrix} = \begin{bmatrix}\dfrac{x_2 - x}{l} & \dfrac{x - x_1}{l}\end{bmatrix} \quad \textbf{\textit{Answer}}$$

	OR	
	b) Determine the shape functions for the Constant Strain Triangle (CST) using polynomial functions.	8

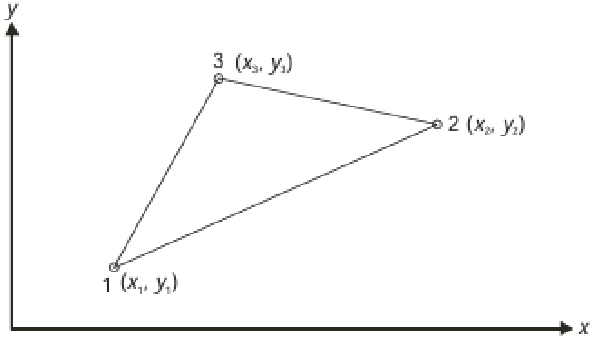


Fig. 5.7

$$\{\delta\}^T = [u_1 \;\; u_2 \;\; u_3 \;\; v_1 \;\; v_2 \;\; v_3]$$

From the consideration of compatibility and completeness the following displacement model is selected

$$u = \alpha_1 + \alpha_2\, x + \alpha_3\, y$$

$$v = \alpha_4 + \alpha_5\, x + \alpha_6\, y \qquad \ldots(5.14)$$

$$\therefore \;\; u_1 = \alpha_1 + \alpha_2\, x_1 + \alpha_3\, y_1$$

$$u_2 = \alpha_1 + \alpha_2\, x_2 + \alpha_3\, y_2$$

$$u_3 = \alpha_1 + \alpha_2\, x_3 + \alpha_3\, y_3$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

$$\therefore \;\; \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Now

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 2A$$

Where A is the area of triangle with vertices at (x_1, y_1), (x_2, y_2) and (x_3, y_3) i.e., the area of the element.

$$\therefore \;\; \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}^T \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A}\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}^T \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

where $a_1 = x_2 y_3 - x_3 y_2$ $\quad a_2 = x_3 y_1 - x_1 y_3$ $\quad a_3 = x_1 y_2 - x_2 y_1$

$b_1 = y_2 - y_3$ $\quad b_2 = y_3 - y_1$ $\quad b_3 = y_1 - y_2$

$c_1 = x_3 - x_2$ $\quad c_2 = x_1 - x_3$ $\quad c_3 = x_2 - x_1$,

same as used in deriving natural coordinates.

$$\therefore\ u = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$= \begin{bmatrix} 1 & x & y \end{bmatrix}\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x & y \end{bmatrix}\frac{1}{2A}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{bmatrix} \dfrac{a_1 + b_1 x + c_1 y}{2A} & \dfrac{a_2 + b_2 x + c_2 y}{2A} & \dfrac{a_3 + b_3 x + c_3 y}{2A} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [N]\{\delta\}_e$$

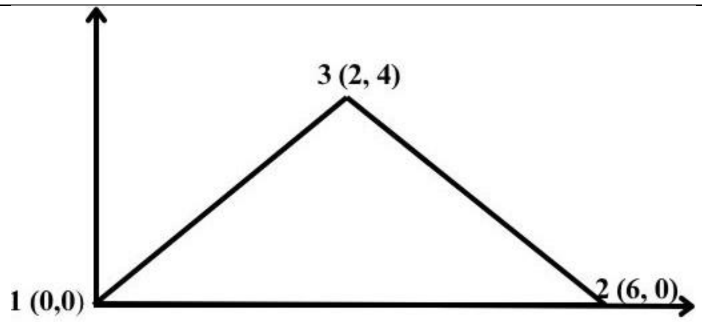
where $\quad [N] = [N_1 \ \ N_2 \ \ N_3]$

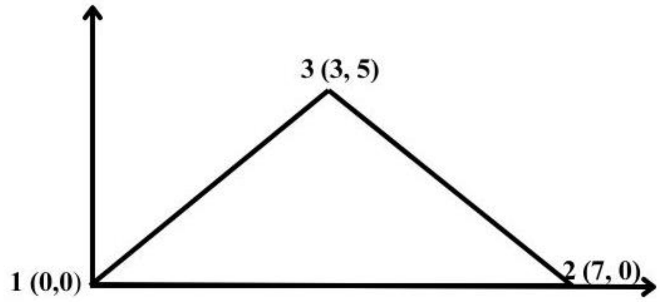
and $\quad N_1 = \dfrac{a_1 + b_1 x + c_1 y}{2A} \quad N_2 = \dfrac{a_2 + b_2 x + c_2 y}{2A}$ and $N_3 = \dfrac{a_3 + b_3 x + c_3 y}{2A}$

Similarly $\quad v = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$

$$\therefore\ u(x, y) = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}\{\delta\}_e$$

c) A three nodded triangular element as shown in fig. 3(c) is used in plane elasticity problem. Find shape functions. (8) 8





$(x_1, y_1) = (0,0), (x_2, y_2) = (7,0), (x_3, y_3) = (3,5)$

$$[N] = [P][A]^{-1} \dots\dots\dots (1)$$

$$[N] = [1 \quad x \quad y] \begin{bmatrix} 1 & 0 & 0 \\ 1 & 7 & 0 \\ 1 & 3 & 5 \end{bmatrix}^{-1} \quad \dots\dots\dots..(1)$$

$$[N] = [1 \quad x \quad y] \frac{1}{35} \begin{bmatrix} 35 & 0 & 0 \\ -5 & 5 & 0 \\ -4 & -3 & 7 \end{bmatrix} \dots\dots\dots\dots (3)$$

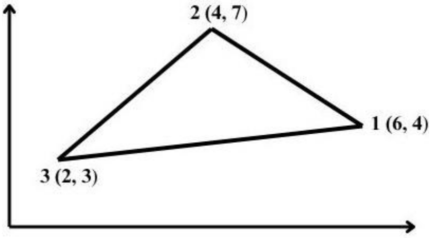
$$N_1 = \frac{35 - 5x - 4y}{35} \dots\dots..(1)$$

$$N_2 = \frac{5x - 3y}{35} \dots\dots..(1)$$

$$N_3 = \frac{7y}{35} \dots\dots..(1)$$

OR

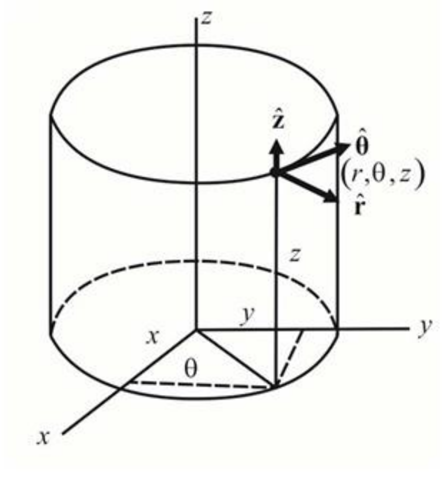
d) Coordinates of nodes of CST are shown in fig. 3(d). At interior point P if x = 2.8 and value of $N_1 = 0.3$, Find coordinate of point P and values of N_2 and N_3. 8



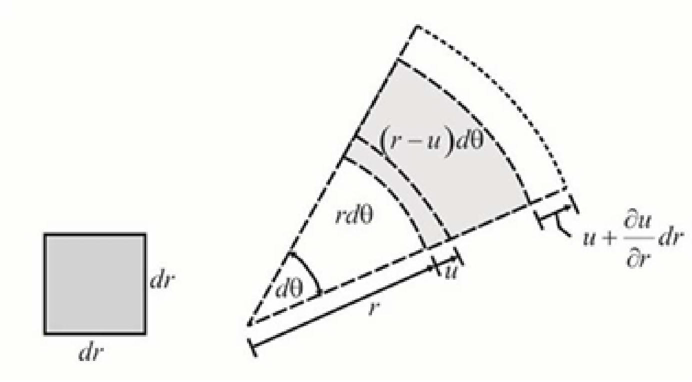
$$(x_1, y_1) = (6,4), (x_2, y_2) = (4,7), (x_3, y_3) = (2,3)$$

$$[N] = [P][A]^{-1} \dots\dots\dots (1)$$

$$[N] = [1 \quad x \quad y] \begin{bmatrix} 1 & 6 & 4 \\ 1 & 4 & 7 \\ 1 & 2 & 3 \end{bmatrix}^{-1} \quad \dots\dots\dots..(1)$$

	$[N] = [1 \quad x \quad y]\frac{1}{14}\begin{bmatrix}3 & -10 & 26\\ 4 & -1 & -3\\ -2 & 4 & -2\end{bmatrix} \ldots\ldots\ldots\ldots(3)$ Putting value of $N_1 = 0.3\ and\ x = 2.8$ $N_1 = \frac{3+4x-2y}{14}; 0.3 = \frac{3+(4\times 2.8)-2y}{14}$ $y = 5.0 \ldots\ldots\ldots(1)$ Using $x = 2.8\ and\ y = 5.0$ $N_2 = \frac{-10-x+4y}{14} = \frac{-10-2.8+(4\times 5.0)}{14} = 0.51 \ldots\ldots..(1)$ $N_3 = \frac{26-3\times 2.8-(2\times 5.0)}{14} = 0.54 \ldots\ldots..(1)$	
4	a) For an axisymmetric element state relation between Strain and Displacement. (8) **5.6.2 Relation between Strain and Displacement** An axisymmetric problem is readily described in cylindrical polar coordinate system: r, z and θ. Here, θ measures the angle between the plane containing the point and the axis of the coordinate system. At $\theta = 0$, the radial and axial coordinates coincide with the global Cartesian X and Y coordinates. Fig. 5.6.2 shows a cylindrical coordinate system and the definition of the position vectors. Let $\hat{r}, \hat{z}$ and $\hat{\theta}$ be unit vectors in the radial, axial, and circumferential directions at a point in the cylindrical coordinate system. system. At $\theta = 0$, the radial and axial coordinates coincide with the global Cartesian X and Y coordinates. Fig. 5.6.2 shows a cylindrical coordinate system and the definition of the position vectors. Let $\hat{r}, \hat{z}$ and $\hat{\theta}$ be unit vectors in the radial, axial, and circumferential directions at a point in the cylindrical coordinate system.  **Fig. 5.6.2 Cylindrical Coordinate System**	8

If the loading consists of radial and axial components that are independent of θ and the material is either isotropic or orthotropic and the material properties are independent of θ, the displacement at any point will only have radial (u_r) and axial (u_z) components. The only stress components that will be nonzero are $\sigma_{rr},\ \sigma_{zz},\ \sigma_{\theta\theta}\ and\ \tau_{rz}$.



(a) Element in r-z plane (b) Element in r-θ plane

A differential element of the body in the *r-z* plane is shown in Fig. 5.6.3(a). The element undergoes deformation in the radial direction. Therefore, it initiates increase in circumference and associated circumferential strain. Let denote the radial displacement as *u*, the circumferential displacement as *v*, and the axial displacement as *w*. Dashed line represents the deformed positions of the body in Fig. 5.6.3(b). The radial strain can be calculated from the above diagram as

$$\varepsilon_r=\frac{1}{dr}\left(u+\frac{\partial u}{\partial r}\times dr-u\right)=\frac{\partial u}{\partial r} \qquad (5.6.1)$$

Since the *rz* plane is effectively the same as a rectangular coordinate system, the axial strain will become

$$\varepsilon_z=\frac{1}{dz}\left(w+\frac{\partial w}{\partial z}\times dz-w\right)=\frac{\partial w}{\partial z} \qquad (5.6.2)$$

Considering the original arc length versus the deformed arc length, the differential element undergoes an expansion in the circumferential direction. Before deformation, let the arc length is assumed as d*s* = *r*dθ. After deformation, the arc length will become d*s* = (*r*+*u*) dθ. Thus, the tangential strain will be

$$\varepsilon_\theta=\frac{(r+u)d\theta-rd\theta}{rd\theta}=\frac{u}{r} \qquad (5.6.3)$$

Similarly, the shear strain will be

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \qquad (5.6.4)$$

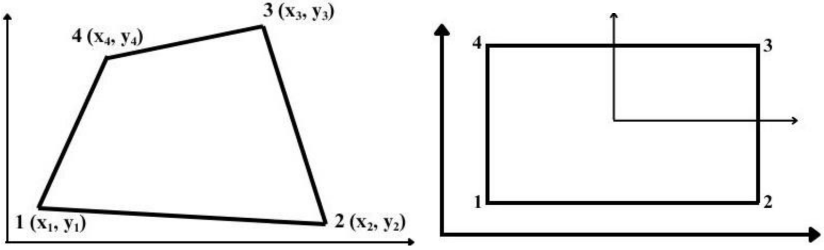
$$\gamma_{r\theta} = 0 \text{ and } \gamma_{z\theta} = 0$$

Thus, there are four strain components present in this case and is given by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{u}{r} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial r} & 0 \\ 0 & \dfrac{\partial}{\partial z} \\ \dfrac{1}{r} & 0 \\ \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial r} \end{bmatrix} \begin{Bmatrix} u \\ w \end{Bmatrix} \qquad (5.6.5)$$

OR

b) Derive Jacobian matrix for four nodded iso-parametric quadrilateral element. 8



1. **Shape functions**

$$N_1 = \frac{(1-\zeta)(1-\eta)}{4};\ N_2 = \frac{(1+\zeta)(1-\eta)}{4};\ N_3 = \frac{(1+\zeta)(1+\eta)}{4};\ N_4 = \frac{(1-\zeta)(1+\eta)}{4}$$

2. **Displacement**

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$$
$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4$$

3. **Strains**

$$\epsilon_x = \frac{\partial u}{\partial x},\ \epsilon_y = \frac{\partial v}{\partial y},\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\in_x = \frac{\partial N_1}{\partial x}u_1 + \frac{\partial N_2}{\partial x}u_2 + \frac{\partial N_3}{\partial x}u_3 + \frac{\partial N_4}{\partial x}u_4$$

$$\in_y = \frac{\partial N_1}{\partial y}v_1 + \frac{\partial N_2}{\partial y}v_2 + \frac{\partial N_3}{\partial y}v_3 + \frac{\partial N_4}{\partial y}v_4$$

$$\gamma_{xy} = \frac{\partial N_1}{\partial y}u_1 + \frac{\partial N_2}{\partial y}u_2 + \frac{\partial N_3}{\partial y}u_3 + \frac{\partial N_4}{\partial y}u_4 + \frac{\partial N_1}{\partial x}v_1 + \frac{\partial N_2}{\partial x}v_2 + \frac{\partial N_3}{\partial x}v_3 + \frac{\partial N_4}{\partial x}v_4$$

1. **Jacobian Matrix**

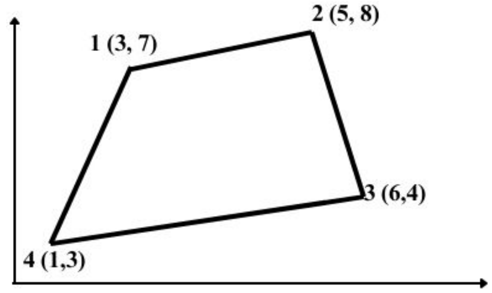
Appling chain rule to represent derivatives of shape functions w.r.t. cartesian coordinates

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial x};\ \frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \zeta}\frac{\partial \zeta}{\partial y} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \zeta}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \zeta}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \zeta} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \zeta} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \zeta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

c) Determine Jacobian Matrix for four nodded iso-parametric quadrilateral element as shown in fig. 4c (8) 8



1. Parent element-
2. Shape function-

$$N_1 = \frac{(1-\zeta)(1+\eta)}{4}; N_2 = \frac{(1+\zeta)(1+\eta)}{4}; N_3 = \frac{(1+\zeta)(1-\eta)}{4}; N_4 = \frac{(1-\zeta)(1-\eta)}{4}$$

$$Geometry - cartesian\ cordinates$$

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$

3. Jacobian Matrix

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \zeta} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \zeta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\frac{\partial x}{\partial \zeta} = \frac{\partial N_1}{\partial \zeta}x_1 + \frac{\partial N_2}{\partial \zeta}x_2 + \frac{\partial N_3}{\partial \zeta}x_3 + \frac{\partial N_4}{\partial \zeta}x_4$$

$$\frac{\partial x}{\partial \eta} = \frac{\partial N_1}{\partial \eta}x_1 + \frac{\partial N_2}{\partial \eta}x_2 + \frac{\partial N_3}{\partial \eta}x_3 + \frac{\partial N_4}{\partial \eta}x_4$$

$$\frac{\partial y}{\partial \zeta} = \frac{\partial N_1}{\partial \zeta}y_1 + \frac{\partial N_2}{\partial \zeta}y_2 + \frac{\partial N_3}{\partial \zeta}y_3 + \frac{\partial N_4}{\partial \zeta}y_4$$

4. $\frac{\partial y}{\partial \eta} = \frac{\partial N_1}{\partial \zeta}y_1 + \frac{\partial N_2}{\partial \eta}y_2 + \frac{\partial N_3}{\partial \eta}y_3 + \frac{\partial N_4}{\partial \eta}y_4$Parent element-
5. Shape function-

$$N_1 = \frac{(1-\zeta)(1+\eta)}{4}; N_2 = \frac{(1+\zeta)(1+\eta)}{4}; N_3 = \frac{(1+\zeta)(1-\eta)}{4}; N_4 = \frac{(1-\zeta)(1-\eta)}{4}$$

$$Geometry - cartesian\ cordinates$$

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$

6. Jacobian Matrix

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \zeta} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \zeta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\frac{\partial x}{\partial \zeta} = \frac{\partial N_1}{\partial \zeta} x_1 + \frac{\partial N_2}{\partial \zeta} x_2 + \frac{\partial N_3}{\partial \zeta} x_3 + \frac{\partial N_4}{\partial \zeta} x_4$$

$$\frac{\partial x}{\partial \eta} = \frac{\partial N_1}{\partial \eta} x_1 + \frac{\partial N_2}{\partial \eta} x_2 + \frac{\partial N_3}{\partial \eta} x_3 + \frac{\partial N_4}{\partial \eta} x_4$$

$$\frac{\partial y}{\partial \zeta} = \frac{\partial N_1}{\partial \zeta} y_1 + \frac{\partial N_2}{\partial \zeta} y_2 + \frac{\partial N_3}{\partial \zeta} y_3 + \frac{\partial N_4}{\partial \zeta} y_4$$

$$\frac{\partial y}{\partial \eta} = \frac{\partial N_1}{\partial \zeta} y_1 + \frac{\partial N_2}{\partial \eta} y_2 + \frac{\partial N_3}{\partial \eta} y_3 + \frac{\partial N_4}{\partial \eta} y_4$$

$$\frac{\partial N_1}{\partial \zeta} = \frac{-1(1+\eta)}{4}; \frac{\partial N_2}{\partial \zeta} = \frac{(1+\eta)}{4}; \frac{\partial N_3}{\partial \zeta} = \frac{(1-\eta)}{4}; \frac{\partial N_4}{\partial \zeta} = \frac{-1(1-\eta)}{4}$$

$$\frac{\partial N_1}{\partial \eta} = \frac{(1-\zeta)}{4}; \frac{\partial N_2}{\partial \eta} = \frac{(1+\zeta)}{4}; \frac{\partial N_3}{\partial \eta} = \frac{-1(1+\zeta)}{4}; \frac{\partial N_4}{\partial \eta} = \frac{-1(1-\zeta)}{4}$$

Putting values of cartesian coordinates

$$\frac{\partial x}{\partial \zeta} = \frac{-1(1+\eta)}{4}(3) + \frac{(1+\eta)}{4}(5) + \frac{(1-\eta)}{4}(6) + \frac{-1(1-\eta)}{4}(1)$$

$$\frac{\partial x}{\partial \zeta} = \frac{-3-3\eta+5+5\eta+6-6\eta-1+1\eta}{4} = \frac{7-3\eta}{4}$$

$$\frac{\partial x}{\partial \eta} = \frac{(1-\zeta)}{4}(3) + \frac{(1+\zeta)}{4}(5) + \frac{-1(1+\zeta)}{4}(6) + \frac{-1(1-\zeta)}{4}(1)$$

$$\frac{\partial x}{\partial \eta} = \frac{3-3\zeta+5+5\zeta-6-6\zeta-1+1\zeta}{4} = \frac{1+3\zeta}{4}$$

$$\frac{\partial y}{\partial \zeta} = \frac{-1(1+\eta)}{4}(7) + \frac{(1+\eta)}{4}(8) + \frac{(1-\eta)}{4}(4) + \frac{-1(1-\eta)}{4}(3)$$

$$\frac{\partial y}{\partial \zeta} = \frac{-7-7\eta+8+8\eta+4-4\eta-3+3\eta}{4} = \frac{2}{4}$$

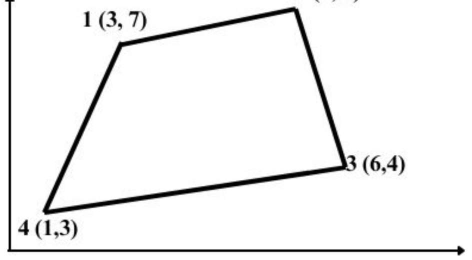
$$\frac{\partial y}{\partial \eta} = \frac{(1-\zeta)}{4}(7) + \frac{(1+\zeta)}{4}(8) + \frac{-1(1+\zeta)}{4}(4) + \frac{-1(1-\zeta)}{4}(3)$$

$$\frac{\partial y}{\partial \eta} = \frac{7-7\zeta+8+8\zeta-4-4\zeta-3+3\zeta}{4} = \frac{8}{4}$$

$$[J] = \begin{bmatrix} \frac{7-3\eta}{4} & \frac{1+3\zeta}{4} \\ \frac{2}{4} & \frac{8}{4} \end{bmatrix}$$

OR

d) Determine natural coordinates (ζ, η) of the any point P whose cartesian coordinates are (3, 4) for four nodded iso-parametric quadrilateral elements as shown in fig. 4(d) — 8

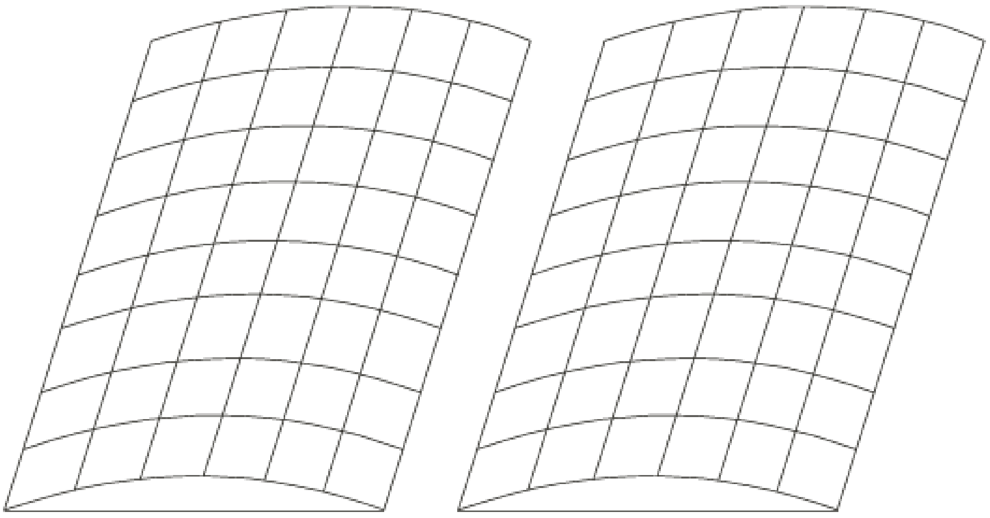


1. Shape function-

	$N_1 = \frac{(1-\zeta)(1+\eta)}{4}; N_2 = \frac{(1+\zeta)(1+\eta)}{4}; N_3 = \frac{(1+\zeta)(1-\eta)}{4}; N_4 = \frac{(1-\zeta)(1-\eta)}{4}$ $Geometry - cartesian\ cordinates$ $x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$ $y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$ For x = 3 and y = 4, $3 = \frac{(1-\zeta)(1+\eta)}{4}(3) + \frac{(1+\zeta)(1+\eta)}{4}(5) + \frac{(1+\zeta)(1-\eta)}{4}(6) + \frac{(1-\zeta)(1-\eta)}{4}(1)$ $3 = \frac{3(1+\eta-\zeta-\zeta\eta) + 5(1+\eta+\zeta+\zeta\eta) + 6(1-\eta+\zeta-\zeta\eta) + 1(1-\eta-\zeta+\zeta\eta)}{4}$ $12 = 2 + 2\eta - 2\zeta - 2\zeta\eta + 3 + 3\eta + 3\zeta + 3\zeta\eta + 5 - 5\eta + 5\zeta - 5\zeta\eta + 1 - 1\eta - 1\zeta + 1\zeta\eta$ $12 = 11 - 1\eta + 5\zeta + 5\zeta\eta, 1 = -1\eta + 5\zeta - 5\zeta\eta \ldots\ldots.. eq. 1$ $\eta =$ $4 = \frac{(1-\zeta)(1+\eta)}{4}(7) + \frac{(1+\zeta)(1+\eta)}{4}(8) + \frac{(1+\zeta)(1-\eta)}{4}(4) + \frac{(1-\zeta)(1-\eta)}{4}(3)$ $4 = \frac{7(1+\eta-\zeta-\zeta\eta) + 8(1+\eta+\zeta+\zeta\eta) + 4(1-\eta+\zeta-\zeta\eta) + 3(1-\eta-\zeta+\zeta\eta)}{4}$ $16 = 5 + 5\eta - 5\zeta - 5\zeta\eta + 6 + 6\eta + 6\zeta + 6\zeta\eta + 3 - 3\eta + 3\zeta - 3\zeta\eta + 2 - 2\eta - 2\zeta + 2\zeta\eta$ $16 = 17 + 7\eta + 3\zeta + \zeta\eta, -1 = 7\eta + 3\zeta + \zeta\eta \ldots\ldots.. eq. 2$ Putting η =, $-1 = (7 \times 3) + 3\zeta + 3\zeta, -22 = 6\zeta,$ $\zeta, \eta =$	
5	a) Explain with neat sketches the various three-dimensional elements used in the analysis of shells. (8) 1. Flat Elements 2. Curved Elements 3. Solid Elements 4. Degenerated Solid Elements. The above elements are briefly explained below and their performance is commented.	8

1. Flat Elements

The earliest method to analysis shells by finite element method was to approximate the curved surface with a number of flat elements. Fig. 16.2 shows approximation of a cylindrical shell roof by a number of flat elements. Since shell, have bending as well as in plane forces, for flat element stiffness matrix should be assembled using both plate bending consideration and considering in plane forces. Fig. 16.3 shows in plane and bending forces to be considered. One can use triangular, rectangular or quadrilateral plate elements. Smaller the element size, better is the result. The development of such shell elements progressed along with the development of plate elements. Using such elements arch dams, cylindrical shell roofs and cooling towers have been successfully analysed by zienkiewicz et al.[3, 4, 5]



(a) Actual shell surface

(b) Approximated shell surface

Fig. 16.2 *Flat elements used for shell analysis*

1. Flat Elements

The earliest method to analysis shells by finite element method was to approximate the curved surface with a number of flat elements. Fig. 16.2 shows approximation of a cylindrical shell roof by a number of flat elements. Since shell, have bending as well as in plane forces, for flat element stiffness matrix should be assembled using both plate bending consideration and considering in plane forces. Fig. 16.3 shows in plane and bending forces to be considered. One can use triangular, rectangular or quadrilateral plate elements. Smaller the element size, better is the result. The development of such shell elements progressed along with the development of plate elements. Using such elements arch dams, cylindrical shell roofs and cooling towers have been successfully analysed by zienkiewicz et al.[3, 4, 5]

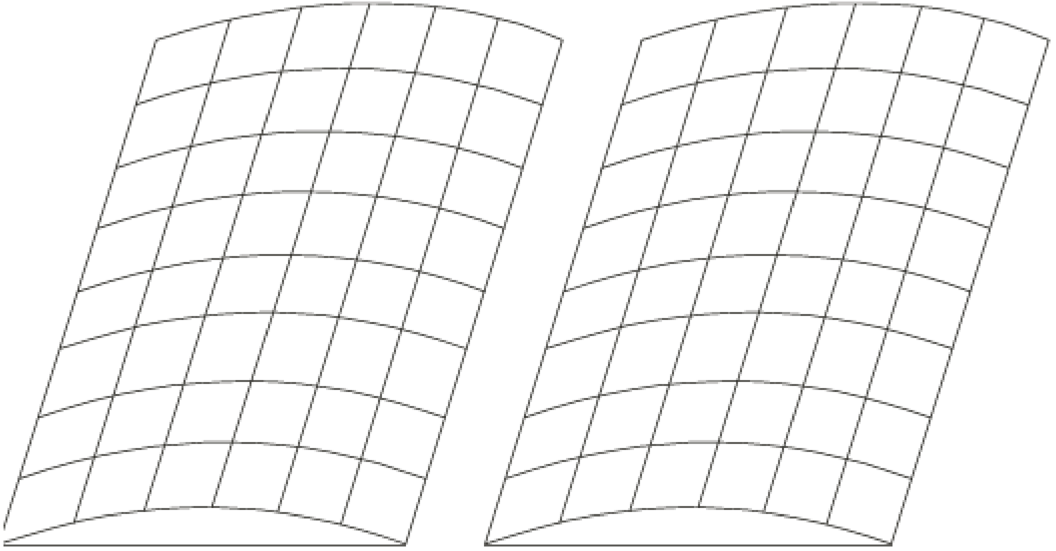
The shortcomings of these flat elements (also called as Facet Elements) are as listed below:

(i) The curvature of the elements is absent within the element.

(ii) The discontinuities of slope between the plate elements produce spurious moments.

(iii) The plate elements themselves have limitations in the analysis of plates, which continues to stay in the shell analysis too.

However singly curved shells may be analysed satisfactorily by taking refined meshes.



(a) Actual shell surface (b) Approximated shell surface

Fig. 16.2 *Flat elements used for shell analysis*

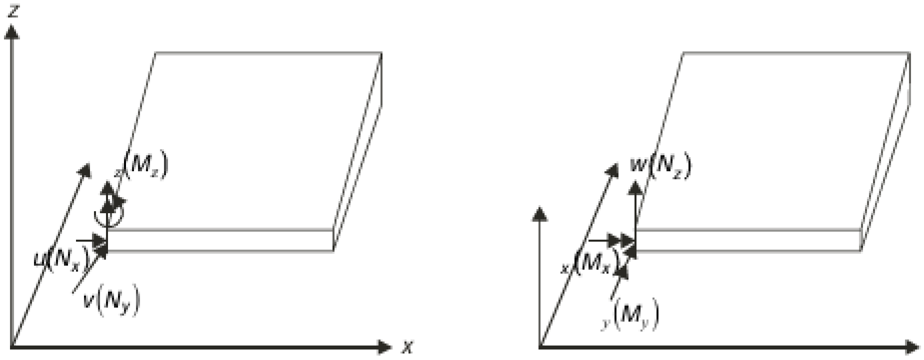


Fig. 16.3 *(a) Inplane forces and deformations (b) Bending forces and deformations*

2. Curved Shell Elements

There are a number of practical problems in which we come across axi-symmetric shell analysis. Fig. 16.4 shows one such case.

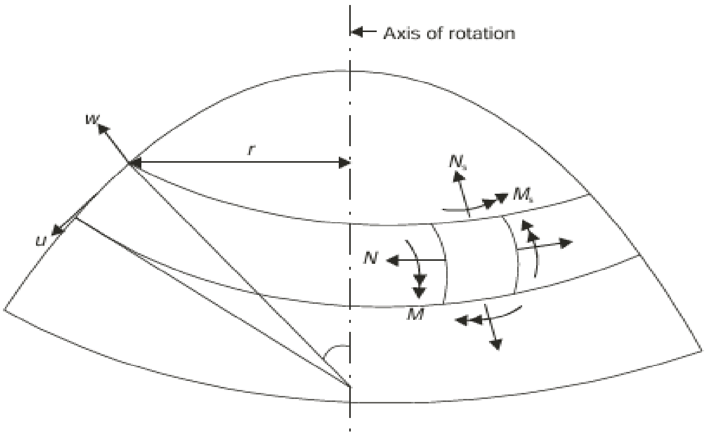
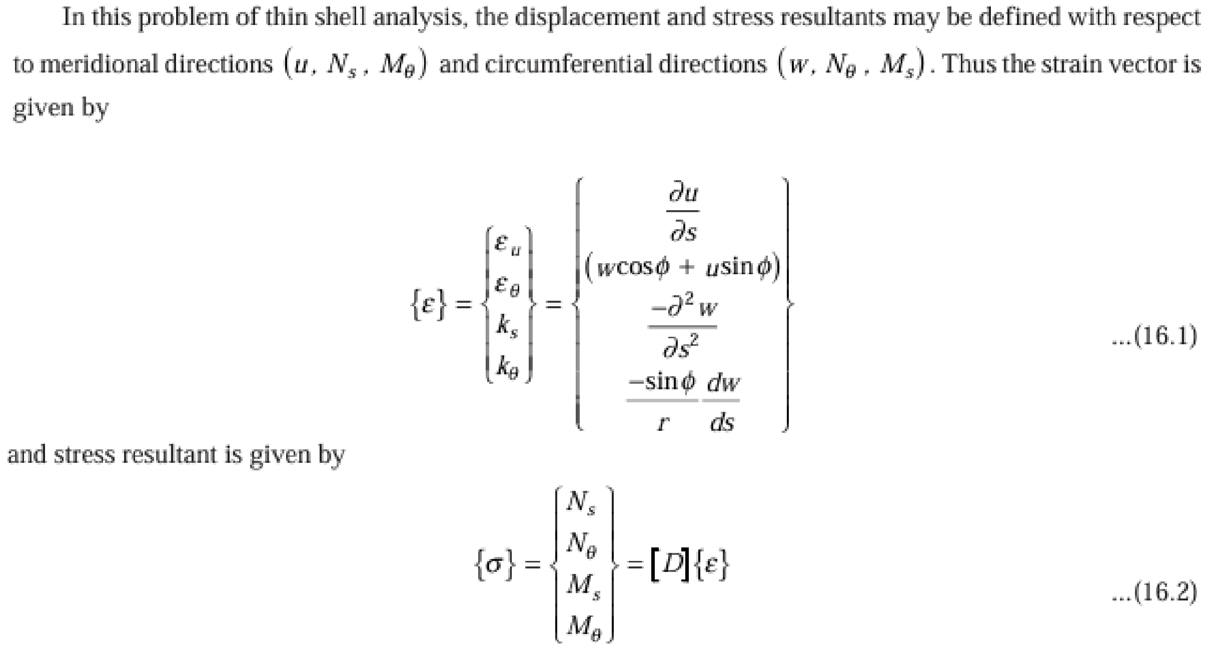


Fig. 16.4 *Axisymmetric shell*

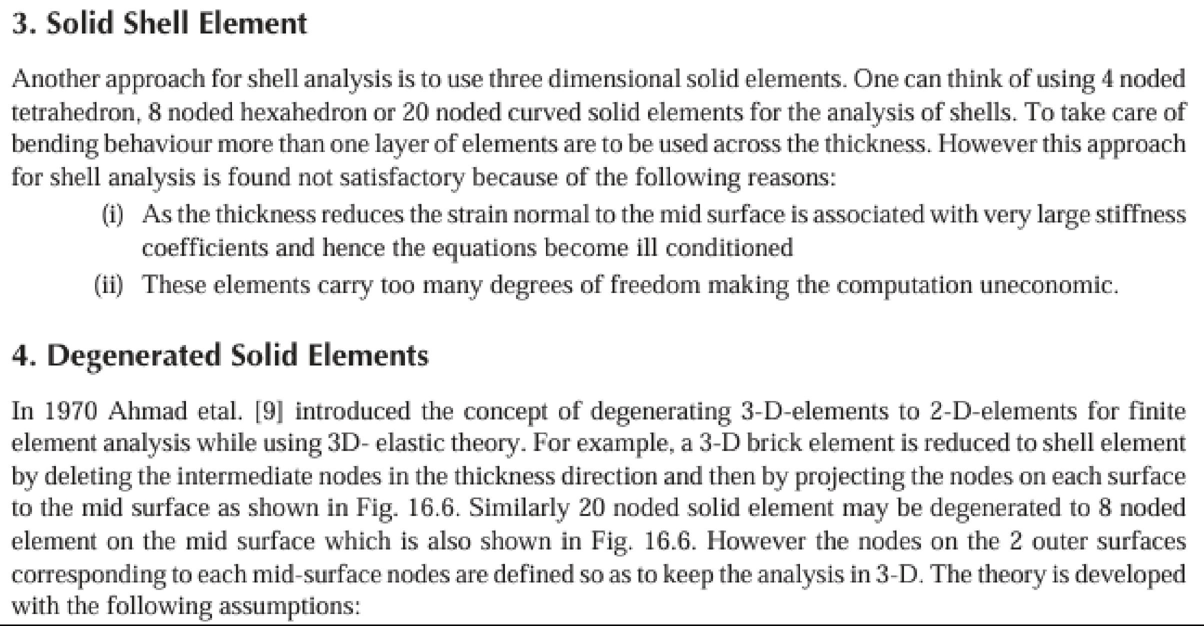


In this problem of thin shell analysis, the displacement and stress resultants may be defined with respect to meridional directions $(u,\ N_s,\ M_\theta)$ and circumferential directions $(w,\ N_\theta,\ M_s)$. Thus the strain vector is given by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_u \\ \varepsilon_\theta \\ k_s \\ k_\theta \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial s} \\ (w\cos\phi + u\sin\phi) \\ \dfrac{-\partial^2 w}{\partial s^2} \\ \dfrac{-\sin\phi}{r}\dfrac{dw}{ds} \end{Bmatrix} \qquad \text{...(16.1)}$$

and stress resultant is given by

$$\{\sigma\} = \begin{Bmatrix} N_s \\ N_\theta \\ M_s \\ M_\theta \end{Bmatrix} = [D]\{\varepsilon\} \qquad \text{...(16.2)}$$



3. Solid Shell Element

Another approach for shell analysis is to use three dimensional solid elements. One can think of using 4 noded tetrahedron, 8 noded hexahedron or 20 noded curved solid elements for the analysis of shells. To take care of bending behaviour more than one layer of elements are to be used across the thickness. However this approach for shell analysis is found not satisfactory because of the following reasons:

(i) As the thickness reduces the strain normal to the mid surface is associated with very large stiffness coefficients and hence the equations become ill conditioned

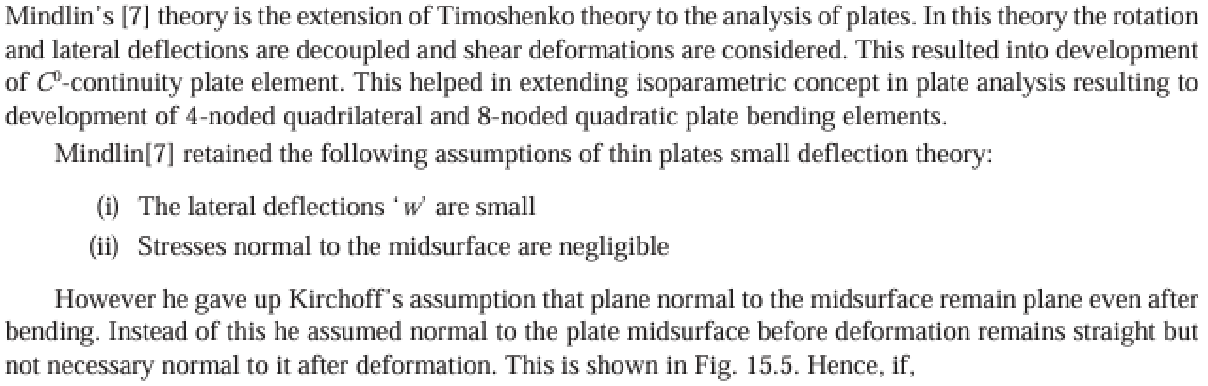
(ii) These elements carry too many degrees of freedom making the computation uneconomic.

4. Degenerated Solid Elements

In 1970 Ahmad etal. [9] introduced the concept of degenerating 3-D-elements to 2-D-elements for finite element analysis while using 3D- elastic theory. For example, a 3-D brick element is reduced to shell element by deleting the intermediate nodes in the thickness direction and then by projecting the nodes on each surface to the mid surface as shown in Fig. 16.6. Similarly 20 noded solid element may be degenerated to 8 noded element on the mid surface which is also shown in Fig. 16.6. However the nodes on the 2 outer surfaces corresponding to each mid-surface nodes are defined so as to keep the analysis in 3-D. The theory is developed with the following assumptions:

OR

b) Explain Mindlin's theory of plate element. 8



Mindlin's [7] theory is the extension of Timoshenko theory to the analysis of plates. In this theory the rotation and lateral deflections are decoupled and shear deformations are considered. This resulted into development of C^0-continuity plate element. This helped in extending isoparametric concept in plate analysis resulting to development of 4-noded quadrilateral and 8-noded quadratic plate bending elements.

Mindlin[7] retained the following assumptions of thin plates small deflection theory:

(i) The lateral deflections 'w' are small

(ii) Stresses normal to the midsurface are negligible

However he gave up Kirchoff's assumption that plane normal to the midsurface remain plane even after bending. Instead of this he assumed normal to the plate midsurface before deformation remains straight but not necessary normal to it after deformation. This is shown in Fig. 15.5. Hence, if,

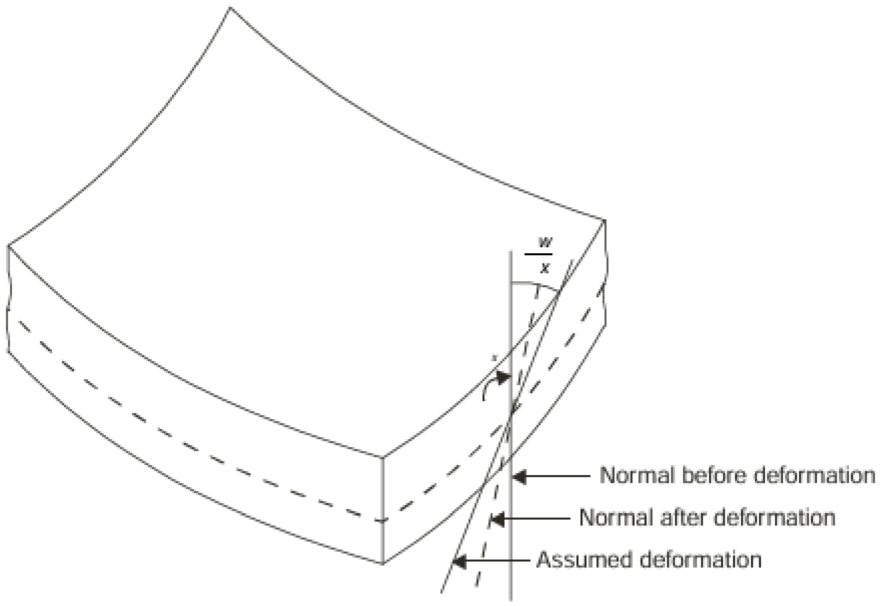


Fig. 15.5 *Deformation of the plate in xz-plane*

θ_x is the final rotation in *x*-direction, we get

$\theta_x = \dfrac{\partial w}{\partial x}$ -average shear strain in *x*-direction i.e.,

$$\theta_x = \frac{\partial w}{\partial x} - \phi_x$$

Similarly

$$\theta_y = \frac{\partial w}{\partial x} - \phi_y \qquad \ldots(15.26)$$

At any node there are three independent field variables w, θ_x and θ_y. The displacement at any point inside the element is given by

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} \qquad \ldots(15.27)$$

where n is the number of nodes in the element. For quadrilateral element $n = 4$ and

$$N_1 = \frac{(1-\xi)(1-\eta)}{4},\ N_2 = \frac{(1+\xi)(1-\eta)}{4}$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4} \text{ and } N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

Similarly for quadratic element $n = 8$, and the shape functions are as given in equation 5.44. For cubic element the shape functions are as presented in equation 5.46. The position of the point itself is given by,

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} \sum N_i x_i \\ \sum N_i y_i \end{Bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \end{Bmatrix} \qquad \ldots(15.28)$$

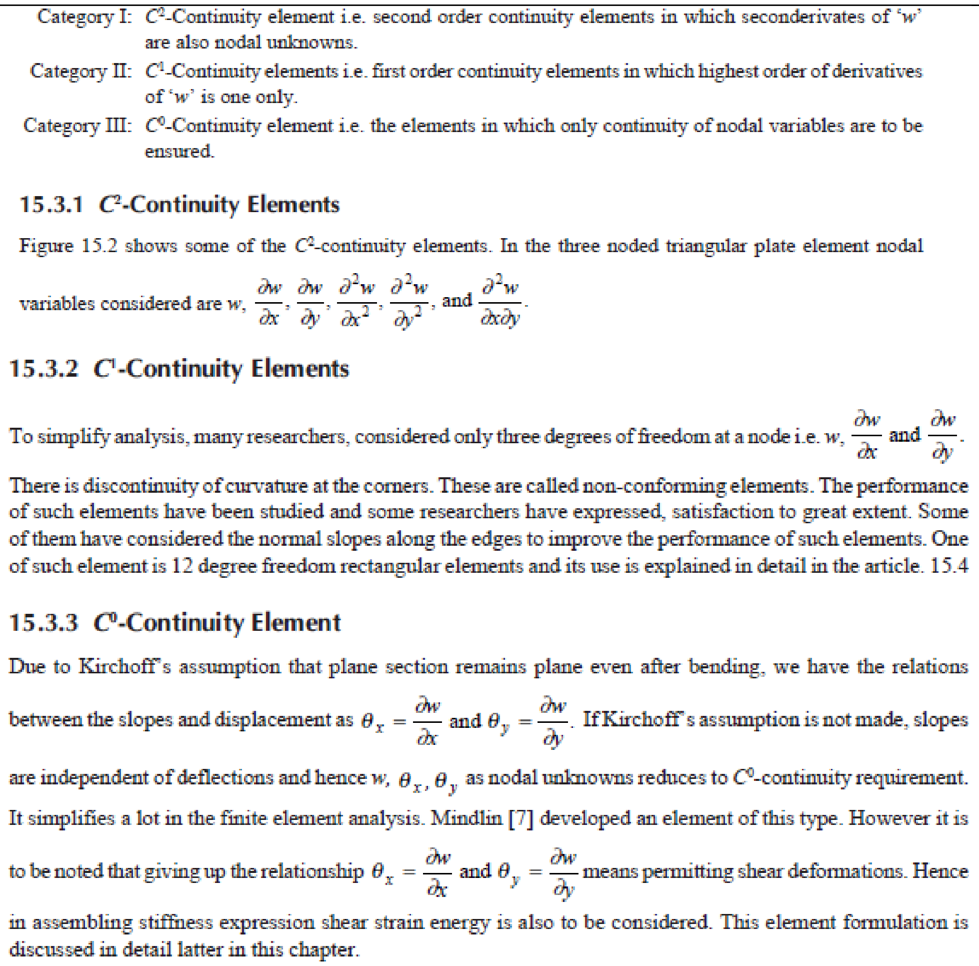
The measure of strain in this element includes both flexural strain k and shear strain ε. The flexural strain is given by.

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} \sum N_i x_i \\ \sum N_i y_i \end{Bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \end{Bmatrix} \qquad \ldots(15.28)$$

The measure of strain in this element includes both flexural strain k and shear strain ε. The flexural strain is given by.

$$\varepsilon_f = \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial y} \\ \dfrac{\partial \theta_y}{\partial x} + \dfrac{\partial \theta_x}{\partial y} \end{Bmatrix} \qquad \ldots(15.29)$$

	c) Continuity	8

	 Category I: C^2-Continuity element i.e. second order continuity elements in which seconderivates of 'w' are also nodal unknowns. Category II: C^1-Continuity elements i.e. first order continuity elements in which highest order of derivatives of 'w' is one only. Category III: C^0-Continuity element i.e. the elements in which only continuity of nodal variables are to be ensured. **15.3.1 C^2-Continuity Elements** Figure 15.2 shows some of the C^2-continuity elements. In the three noded triangular plate element nodal variables considered are w, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial y^2}$, and $\frac{\partial^2 w}{\partial x \partial y}$. **15.3.2 C^1-Continuity Elements** To simplify analysis, many researchers, considered only three degrees of freedom at a node i.e. w, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$. There is discontinuity of curvature at the corners. These are called non-conforming elements. The performance of such elements have been studied and some researchers have expressed, satisfaction to great extent. Some of them have considered the normal slopes along the edges to improve the performance of such elements. One of such element is 12 degree freedom rectangular elements and its use is explained in detail in the article. 15.4 **15.3.3 C^0-Continuity Element** Due to Kirchoff's assumption that plane section remains plane even after bending, we have the relations between the slopes and displacement as $\theta_x = \frac{\partial w}{\partial x}$ and $\theta_y = \frac{\partial w}{\partial y}$. If Kirchoff's assumption is not made, slopes are independent of deflections and hence w, θ_x, θ_y as nodal unknowns reduces to C^0-continuity requirement. It simplifies a lot in the finite element analysis. Mindlin [7] developed an element of this type. However it is to be noted that giving up the relationship $\theta_x = \frac{\partial w}{\partial x}$ and $\theta_y = \frac{\partial w}{\partial y}$ means permitting shear deformations. Hence in assembling stiffness expression shear strain energy is also to be considered. This element formulation is discussed in detail latter in this chapter.	
	OR	
	d) Write displacement fields in 4 nodded degenerated shell elements	8

Let u, v, w be displacement of a point having its local coordinate; $u_i\, v_i\, w_i$ be the displacement of corresponding mid surface which is having local coordinates ξ, η (ref. Fig. 16.8)

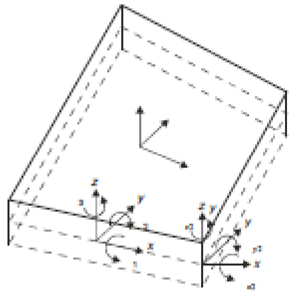


Fig. 16.8 *Displacement field in 4 noded degenerated shell element*

Let u_i^*, v_i^*, w_i^* be the relative displacement along x, y, z directions due to rotation of normal at node i. i.e. $\theta_{xi}, \theta_{yi}, \theta_{zi}$ about the global axes. Then

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^{4} N_i \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \begin{Bmatrix} u_i^* \\ v_i^* \\ w_i^* \end{Bmatrix} \qquad ...(16.9)$$

If $\alpha'_{1i}, \alpha'_{2i}, \alpha'_{3i}$ are the normal rotations at 'i' about axes x', y', z' with the shell assumption of straight normal to middle surface remain straight even after deformation, α'_{3i} becomes zero.

$$\therefore \begin{Bmatrix} u_i' \\ v_i' \\ w_i' \end{Bmatrix} = \zeta \frac{h_i}{2} \begin{Bmatrix} \alpha_{2i} \\ -\alpha_{1i} \\ 0 \end{Bmatrix} \qquad ...(16.10)$$

in which u_i', v_i', w_i' are displacements along axes x', y', z' respectively.

If the direction cosines between global and local axes are l_1, m_1, n_1; $l_2\, m_2, n_2$, l_3, m_3, n_3, then

$$u_i^* = l_{1i}\, u_i' + l_{2i}\, v_i'$$

$$v_i^* = m_{1i}\, u_i' + m_{2i}\, v_i'$$

$$w_i^* = n_{1i}\, u_i' + n_{2i}\, v_i'$$

i.e.,

$$\begin{Bmatrix} u_i^* \\ v_i^* \\ w_i^* \end{Bmatrix} = \begin{bmatrix} l_{1i} & l_{2i} \\ m_{1i} & m_{2i} \\ n_{1i} & n_{2i} \end{bmatrix} \begin{Bmatrix} u_i' \\ v_i' \end{Bmatrix}$$

The rotation α_{1i} and α_{2i} are given by the relation,

$$\begin{Bmatrix} \alpha_{1i} \\ \alpha_{2i} \end{Bmatrix} = \begin{Bmatrix} l_{1i} & m_{1i} & n_{1i} \\ l_{2i} & m_{2i} & n_{2i} \end{Bmatrix} \begin{Bmatrix} \theta_{xi} \\ \theta_{yi} \\ \theta_{zi} \end{Bmatrix}$$

From equation 16.11, 16.10 and 16.12 we get,

$$\begin{Bmatrix} u_i^* \\ v_i^* \\ w_i^* \end{Bmatrix} = \begin{Bmatrix} l_{1i} & l_{2i} \\ m_{1i} & m_{2i} \\ n_{1i} & n_{2i} \end{Bmatrix} \frac{h_i}{2} \zeta \begin{bmatrix} l_{2i} & m_{2i} & n_{2i} \\ -l_{1i} & -m_{2i} & -n_{2i} \end{bmatrix} \begin{Bmatrix} \theta_{xi} \\ \theta_{yi} \\ \theta_{zi} \end{Bmatrix}$$

$$= \zeta \frac{h_i}{2} \begin{bmatrix} 0 & n_{3i} & -m_{3i} \\ -n_{3i} & 0 & l_{3i} \\ m_{3i} & -l_{3i} & 0 \end{bmatrix} \begin{Bmatrix} \theta_{xi} \\ \theta_{yi} \\ \theta_{zi} \end{Bmatrix} = \zeta \frac{h_i}{2} \begin{Bmatrix} n_{3i}\theta_{yi} - m_{3i}\theta_{zi} \\ -n_{3i}\theta_{xi} + l_{3i}\theta_{zi} \\ m_{3i}\theta_{xi} - l_{3i}\theta_{yi} \end{Bmatrix}$$

Substituting equation 16.13 in equation 16.9 we get

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^{4} N_i \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \frac{\zeta h_i}{2} \begin{Bmatrix} n_{3i}\theta_{yi} - m_{3i}\theta_{zi} \\ -n_{3i}\theta_{xi} - l_{3i}\theta_{zi} \\ m_{3i}\theta_{xi} - l_{3i}\theta_{yi} \end{Bmatrix}$$